Graph Theory, Spring 2016, Homework 1

1. Given a graph $G = (V, E, \psi)$, we may define a relation $\sim$ on the edges by defining $e \sim f$ if there is some vertex v incident to e and v is incident to f. Give an example of a graph where this is an equivalence relation. Is this always an equivalence relation? Why or why not?

2. Show that two simple graphs G, H are isomorphic if and only if the following statement is true:

   we may find a bijection $f : V_G \to V_H$ with the property that vertices $v, w \in V_G$ are adjacent in G exactly when $f(v), f(w) \in V_H$ are adjacent in H.

3. Give an example to show that the previous problem is false if G is not simple.

4. Prove that any simple graph with at least 2 vertices must have at least two vertices with the same degree.

5. Prove that if G is a graph, it must have an even number of vertices whose degree is odd.

6. Is it possible to have a simple graph G with 6 vertices whose degrees are $6, 3, 3, 2, 2, 1$? Why or why not? What about a simple graph with 4 vertices whose degrees are $3, 3, 1, 1$?

7. (6000 level) Suppose that G is a simple graph with vertices $v_0, v_1, \ldots, v_n$, of degrees $d_0 \geq d_1 \geq d_2 \geq \cdots \geq d_n$. Show that we may find a new graph G' with vertices $v'_0, v'_1, \ldots, v'_n$ with $deg_G v_i = deg_{G'} v'_i$, and with v'_0 adjacent to the vertices $v'_1, \ldots, v'_{d_0}$.

8. For a simple graph G, we define the complement of G, denoted $\overline{G}$ to be the graph with the same set of vertices (i.e. $V_G = V_{\overline{G}}$) and such that a pair of vertices v, w are adjacent in $\overline{G}$ if and only if they are *not* adjacent in G.

   Show that for every simple graph with 6 vertices, there either exists a subgraph H which is isomorphic to the complete graph K_3, or there exists a subgraph H' which is isomorphic to its complementary graph $\overline{K}_3$.