Graph Theory, Spring 2016, Homework 7

1. Let G be a simple graph. Define a new simple graph G' to have the same vertex set as G, but where v and w are adjacent in G' if and only if either

   - v and w are adjacent in G, or
   - there is some vertex u adjacent to both v and w.

   Now, let $G_1 = G'$, $G_2 = G'_1 = G''$, $G_3 = G'_2$, and in general $G_i = G'_{i-1}$.

   (a) Show that for some i, we have $G_i = G_{i+1} = G_{i+2} = \cdots$.

   (b) Show that $G_n = K_{v(G)}$ for some n if and only if G is connected.

2. Suppose you have a chess board (and 8 by 8 grid) and dominoes, each of which can cover two adjacent squares on the board (in either direction, either up and down, or side to side). Show that it is impossible to cover the board with these dominoes only leaving exactly two opposite corner squares uncovered.

3. Let G be the simple graph whose vertices correspond to the edges of a cube (there are 12 of these), and where two vertices are adjacent if the corresponding edges of the cube intersect at one of the corners of the cube. Does this graph have a Hamiltonian cycle? If so, exhibit it.

4. Let G be a simple graph with no odd length cycles, and such that the maximum degree of any vertex is 4. Show that if G is Hamiltonian (has a Hamiltonian cycle), then G has edge chromatic number 4

5. Show that if a tree T has a perfect matching, then it has exactly one perfect matching.

6. Show that if a graph G has a perfect matching, then for every vertex $v \in V_G$, the graph $G - v$ has exactly one component which has no perfect matching.

7. (8000 level) Show that a tree T has a perfect matching if and only if, for every vertex $v \in V_G$, the graph $T - v$ has exactly one component with an odd number of vertices.