Every Sunday you come to the train station and step onto the first train to arrive, taking it to its first stop. You arrive at a random time each morning.

There are two trains, each running every 10 minutes.

Train A takes you to the Blitz Donut shop

Train B takes you to Martha's Mochi-making shop.

After a while, you realize that you end up at the Mochi Shop about 8 times out of 10. How could this be?

Barber in the village who shaves everyone who doesn't shave themselves.

<u>Q:</u> who shaves the barber?

Sets can sometimes be elements of other sets.

$\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$

Given a set X, it makes sense to ask

is $X \in X$?

let S = the set of all sets that don't contain themselves

$= \{X \mid X \notin X\}$

ex: $\{\emptyset\} \notin \{\emptyset\} \Rightarrow \{\emptyset\} \in S$

$\{1\} \in S$ $\quad 1 \in \{1\}$

$\{1, \{1\}\} \notin \{1, \{1\}\}$

is $S \in S$?

if $S \in S$ then by def. of S it satisfies the property $S \notin S$

So $S \in S$ but $S \notin S$. problem.

if $S \notin S$ then S satisfies the prop. defining S.

so $S \in S$.

"Russel's paradox"

Set theory was proposed (by Cantor, Zermelo, Fraenkel..)
as a new axiomatic framework for math.

Came w/ a formal language

Notion of Sets & rules to talk about them & reason w/ them.

"Axiom of comprehension"

Says: if X is any set, F is a statement about elements of X, true or false for any given $x \in X$ then it makes sense to construct

$$\{x \in X \mid F \text{ is true for } x\}$$

Classic Russel paradox: let $\mathcal{U}$ = ~~the set of all sets~~

$$S = \{X \in \mathcal{U} \mid X \notin X\}$$

List of ~8 Axioms w/out ability to consider

$$S = \{\text{things} \mid \text{stuff}\}$$

"such that"
"satisfying"

"S is the set consisting of the things such that stuff is true"

$\{2n+1 \mid n \in \mathbb{Z}\}$

$= \{m \in \mathbb{Z} \mid m = 2n+1 \text{ for some } n \in \mathbb{Z}\}$

D. Hilbert ~ 1900

Every true statement (in math) should be explanable / justifiable by logical argument.

1621: conjectured every positive integer can be written as the sum of 4 squares.

$1 = 1^2 + 0^2 + 0^2 + 0^2$ $\quad 2 = 1^2 + 1^2 + 0^2 + 0^2$

$11 = 3^2 + 1^2 + 1^2 + 0^2$

$2578 = a^2 + b^2 + c^2 + d^2$

Lagrange 1770 proved this!

Fermat 1637 $\quad a^3 \neq b^3 + c^3$ unless b or c = 0

$8 =$

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$a^4 \neq b^4 + c^4$ $\quad$ $a^5 \neq b^5 + c^5$

I have a remarkable proof of this, but it can't fit in the margin.

"Fermat's last theorem."

Proved in 1995 Andrew Wiles (at Princeton)

Goldbach 1742 any even # greater than or equal to 4 is a sum of 2 primes.

eventually provable?

1931 Gödel

Proved that any consistent formal system able to describe arithmetic of whole numbers has statements which are true (valid) but which cannot be proven within the system.

every formal system is either inconsistent or incomplete.

Idea: encode idea of provability within arithmetic.

Nonetheless: no one has found any inconsistences so far.

On the other hand, we have noticed various statements which are unprovable.

We'll see an example w/ infinite sets soon.

Are we awake?
working with sets.

Infinity

Basic infinite set is $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

We say that infinite sets have the same size if there is a way to put the elements in a 1-to-1 correspondence.

ex: $\#\mathbb{N}$ same as size of $\{-n \mid n \in \mathbb{N}\}$

$\{0, 1, 2, \dots\} \longleftrightarrow \{0, -1, -2, \dots\}$

Bus 1   1   2   3   -   -

Bus 2   1   2   3   -   -

Bus 3   1   2   3   -   -

⋮