Take 2

The participant draws 6 cards from a deck, and hands them to magician A. Magician A then removes one of the cards, declaring it to be the "special card," and hands it back to the participant. Magician A then arranges the remaining 5 cards into a pile and hands them to Magician B. By examining the pile of 5 cards, Magician B is then able to correctly guess what the special card is. How is it possible for them to do this?

"Hint"

By letting some of the card in the pile be face up, the trick can be done with 4 cards instead of 6

Last time: gave lots of examples (via Hilbert hotel) of "countably infinite" sets.

"Cardinality" = "size" of a set

We say two sets X & Y have the same cardinality if we can find a one to one correspondence between their elements.

$\mathbb{N}$ same cardinality as $-\mathbb{N}$

$\{0,1,2,\ldots\} \longleftrightarrow \{0,-1,-2,-3,\ldots\}$

$n \longmapsto -n$

$\{0,1,2,\dots\} \longleftrightarrow \{0,2,4,6,\dots\}$

"$\mathbb{N}$"

$n \longmapsto 2n$

$|X|$ = "the cardinality of X"

$|X|=|Y|$ same cardinality.

$|\mathbb{N}| = \aleph_0$

"$|\mathbb{N}| = \infty$"

$2\aleph_0 = \aleph_0$

$\aleph_0 \cdot \aleph_0 = \aleph_0$

$$\begin{array}{|cccc}
\hline
0 & 1 & 2 & \dots \\
0 & 1 & 2 & \dots \\
0 & 1 & 2 & \dots
\end{array}$$

0 1 2 3 ...

$\sqrt{1}$ $\sqrt{2}$ $\sqrt{3}$

$\aleph_0 + 3 = \aleph_0$

$\aleph_0 \cdot \aleph_0 = \aleph_0$

$\mathbb{R}$ all real #'s.

$[0,1]$

Decimals:

57.2834927719985...

57.2834927729985...

same!
$$\begin{cases} 0.99999\ldots = x \\ 1.0000\ldots \end{cases}$$

5.9344999...

5.9350...

$$10x = 9.999\ldots$$
$$x = 0.999\ldots$$
$$9x = 9 \qquad \underline{x = 1}$$

Dedekind cuts

Claim: $|\mathbb{R}| > \aleph_0 = |\mathbb{N}|$

suppose $|\mathbb{R}| = |\mathbb{N}|$

1 $\quad *.d_1^1 d_2^1 d_3^1 - - -$

2 $\quad *.d_1^2 d_2^2 d_3^2 \cdot$ — —

3 $\quad *.d \cdot -$ —

$\vdots$

One you missed:

$0.101\,10011$

make a 1 if $d_1^1 \neq 1$

. . . 0 else.

Cantor's diagonalization

$$|\mathbb{N}| < \aleph_1 < |\mathbb{R}|$$

"continuum hypothesis"

shown to be independent of axioms by Cohen.

Proofs (argumentations/justifications)

Claim: Given sets A, B, we always have

$$A \cap B \subset A$$

Proof: To show $A \cap B$ is a subset of A, we need to show every element of $A \cap B$ is also an element of A.

Let x be an element of $A \cap B$.
Since $x \in A \cap B$ by definition of the intersection,
it is in both A and B. Therefore it is in A.

So every element of $A \cap B$ is also an element of A
as we wanted to show.

<u>Claim</u> for all sets $A, B,$ $\cup$ = or $\cap$ = and
$\subset$ then.

$$A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$$

Want to show that if $x \in A \cup (B \cap C)$
then $x \in (A \cup B) \cap (A \cup C)$.

So, suppose $x \in A \cup (B \cap C)$. then either $x \in A$
or $x \in B \cap C$

in either case, need to show $x \in (A \cup B) \cap (A \cup C)$

Case 1 if $x \in A$ then since $A \subset A \cup B$, $x \in A \cup B$
since $A \subset A \cup C$, $x \in A \cup C$ so $x \in (A \cup B) \cap (A \cup C)$ ✓

Case 2 if $x \in B \cap C$ then $x \in B$ $B \subset A \cup B$ so
$x \in A \cup B$
and $x \in C$, $C \subset A \cup C$ so $x \in A \cup C$
so $x \in (A \cup B) \cap (A \cup C)$ ✓

Show that

$$X \cap (X \cup Y) \supset X$$

Want to show X is a subset of $X \cap (X \cup Y)$

In other words we want to show every element of X is also an element of $X \cap (X \cup Y)$

Suppose $x \in X$ want to show $x \in X \cap (X \cup Y)$

so want to show by def of $\cap$, $x \in X$ and $x \in X \cup Y$.

By assumption $x \in X$.

So just need to show $x \in X \cup Y$.

but $x \in X$ & $X \subset X \cup Y$ so $x \in X \cup Y$ ✓