Mathematical Statistics

Lecture 1: Probability review, part 1

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Basic objects of probability:

Sample space S $\longleftarrow$ (results of some experiment)

Probability P assigns a real number in $[0,1]$ to certain subsets $E \subset S$ $\quad P(E)$

Subsets $E \subset S$ $\longleftrightarrow$ "event" = collection of potential outcomes

$P(E)$ = probability of obtaining an outcome in E

Axioms

$P(S) = 1 \qquad P(\emptyset) = 0$

if $E_i \subset S \qquad E_i \cap E_j = \emptyset$ (mutually exclusive)

then $P\left(\bigcup_i E_i\right) = \sum_i P(E_i)$

that's it.

Useful notion:

we say $E, F \subset S$ are <u>independent</u>

if $P(E \cap F) = P(E)P(F)$

Random variables

Def A random variable is a function $X : S \to \mathbb{R}$

ex: S = person chosen at random X = ht of a person

Notation: $P(X \leq x) = P(\{s \in S \mid X(s) \leq x\})$

similarly $P(x \leq X)$ $P(a \leq X \leq b)$ etc.

Functions of a random variable

Defi: if $g: \mathbb{R} \to \mathbb{R}$ we write $g(X)$ for the new random variable $g \circ X$

i.e.

$$\begin{tikzcd} S \arrow[r, "X"] \arrow[rr, bend right, "g(X)"'] & \mathbb{R} \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

How to encode the information of a random variable?

Cumulative distribution function (c.d.f.): $F(x) \equiv P(X \leq x)$

(Discrete) Probability distribution: $p(x) \equiv P(X = x)$

(Continuous) Probability density function (p.d.f.) is an $f(x)$

such that $P(a \leq X \leq b) = \int_a^b f(x)\,dx$

(Other way to encode: Moments)

First: Expectation

$$E[X] \equiv \sum x p(x) \quad \text{discrete case}$$

$$\int x f(x) dx \quad \text{continuous case}$$

Property: $E[X+Y] = E[X] + E[Y]$

$$E[\lambda X] = \lambda E[X]$$

Expectation of fcns of random var.

Fact: $E[g(X)] = \sum g(x)f(x)$ discrete

$= \int g(x)f(x)\,dx$ continuous

Moments

Def the i^{th} moment of X about a is

$$E[(X-a)^i]$$

ex: $X \rightsquigarrow f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$

0^{th} moment anywhere $= E[(X-a)^0] = E[1] = 1$

1^{st} moment about $0 = E[X] = \int_0^1 x \cdot 1 = \frac{1}{2}$

2^{nd} moment about $1 = E[(X-1)^2] = \int_0^1 (x-1)^2 \, dx = \dots = \frac{1}{3}$

Moments encode <u>very</u> important info

<u>Def</u> μ = mean = $E[X]$

<u>Def</u> σ^2 = variance = $E[(X-\mu)^2]$

<u>Def</u> μ_i = ith moment about the mean $(\mu_2 = \sigma^2)$

<u>Def</u> μ_i' = ith moment about 0 $(\mu_1' = \mu = E[X])$

Fact: (Usually), X is determined by its moments about 0

$\mu_0', \mu_1', \mu_2', \ldots$ a list of #s.

Useful to "package" these into a power series

$$M_X(t) = \mu_0' + \mu_1' t + \mu_2' \frac{t^2}{2} + \mu_3' \frac{t^3}{3!} + \mu_4' \frac{t^4}{4!} + \cdots$$

"moment generating function"

So $\mu_i' = \frac{d^i}{dt^i} M_X(t) \Big|_{t=0}$.

$\Longrightarrow$ X determined by $M_X(t)$.

Alternate description of $M_X(t) = E[e^{tX}]$

$$\boxed{\text{Nice property} \quad M_{X+Y}(t) = M_X(t)\, M_Y(t)}$$

Very useful tool for identify random vars.

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