normal population

suppose σ^2 unknown, test hypothesis $\mu = \mu_0$

$H_0: \mu = \mu_0$

$H_1: \mu \neq \mu_0$

$$f(x_1, \ldots, x_n) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_i (x_i - \mu)^2}$$

$$= f(x_1, \ldots, x_n; \sigma^2, \mu)$$

$$= L(\sigma^2, \mu; x_1, \ldots, x_n)$$

$$\lambda = \frac{\max_{\sigma^2, \mu \text{ s.t. } H_0} L(\sigma^2, \mu; x)}{\max_{\text{all } \sigma^2, \mu} L(\sigma^2, \mu; x)}$$

$$L(\sigma^2, \mu; x) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_i (x_i - \mu)^2}$$

hold σ^2 fixed, let μ vary.

max $L(\ )$ happen at max $\ln(L(\ ))$

$$\ln L(\sigma^2, \mu; x) = -\frac{n}{2}\left[\ln 2\pi + \ln \sigma^2\right] - \frac{1}{2\sigma^2}\sum_i (x_i - \mu)^2$$

$$\frac{\partial}{\partial \mu}(\ ) = -\frac{1}{\sigma^2}\sum_i (x_i - \mu)(-1) = 0$$

$$\sum x_i = \sum \mu = n\mu$$

$$\mu = \frac{1}{n}\sum x_i = \bar{x}$$

max at $\mu = \bar{x}$

$$\frac{\partial}{\partial \sigma^2}(\quad) = \frac{\partial}{\partial \sigma^2}\left(-\frac{n}{2}\left[\ln 2\pi + \ln \sigma^2\right] - \frac{(\sigma^2)^{-1}}{2}\sum (x_i-\mu)^2\right.$$

$$= -\frac{n}{2}\left(\frac{1}{\sigma^2}\right) + \frac{1}{2\sigma^4}\sum (x_i-\mu)^2 = 0$$

$$\sum (x_i-\mu)^2 = n\sigma^2$$

MLE for σ^2 $\quad \hat{\sigma}^2 = \frac{\sum (x_i-\mu)^2}{n} = \sigma^2$

$$L(\sigma^2, \mu; x) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum (x_i-\mu)^2}$$

when μ varies, max happen at $\mu = \bar{x}$

when μ fixed, σ^2 varies, max at $\sigma^2 = \frac{\sum (x_i-\mu)^2}{n}$

$$\lambda = \frac{\max_{\mu=\mu_0, \sigma^2>0} L(\sigma^2, \mu; x)}{\max_{\mu, \sigma^2>0} L(\sigma^2, \mu; x)} = \frac{L\left(\frac{\sum (x_i-\mu_0)^2}{n}, \mu_0; x\right)}{L\left(\frac{\sum (x_i-\bar{x})^2}{n}, \bar{x}; x\right)}$$

$$\lambda = \frac{\left(\frac{1}{2\pi\left(\frac{\sum(x_i-\mu_0)^2}{n}\right)}\right)^{n/2} e^{-\frac{1}{2\left(\frac{\sum(x_i-\mu_0)^2}{n}\right)}\sum(x_i-\mu_0)^2}}{\left(\frac{1}{2\pi\left(\frac{\sum(x_i-\bar{x})^2}{n}\right)}\right)^{n/2} e^{-\frac{1}{2\left(\frac{\sum(x_i-\bar{x})^2}{n}\right)}\sum(x_i-\bar{x})^2}}$$

$$\lambda = \left(\frac{2\pi\left(\sum(x_i-\bar{x})^2\right)}{2\pi\left(\sum(x_i-\mu_0)^2\right)}\right)^{n/2} \frac{e^{-\frac{n}{2}}}{e^{-n/2}}$$

$$= \left(\frac{\sum(x_i-\bar{x})^2}{\sum(x_i-\mu_0)^2}\right)^{n/2} = \left(\frac{\sum(x_i-\bar{x})^2}{\sum(x_i-\bar{x})^2 + n(\bar{x}-\mu_0)^2}\right)^{n/2}$$

$$\sum(x_i-\bar{x})^2 - \sum(x_i-\mu)^2 = n(\bar{x}-\mu_0)^2$$

{ alg.

dist, gp terms,

use $\sum x_i = n\bar{x}$

$$\lambda = \left(\frac{1}{\frac{\sum(x_i-\bar{x})^2 + n(\bar{x}-\mu_0)^2}{\sum(x_i-\bar{x})^2}}\right) = \frac{1}{1 + \frac{n(\bar{x}-\mu_0)^2}{\sum(x_i-\bar{x})^2}}$$

$$\lambda = \frac{1}{1 + \left(\frac{n}{n-1}\right) \dfrac{(\bar{x}-\mu_0)^2}{\left(\frac{\sum (x_i - \bar{x})^2}{n-1}\right)}} = \frac{1}{1 + \left(\frac{n}{n-1}\right) \frac{(\bar{x}-\mu_0)^2}{s^2}}$$

$$\lambda = \frac{1}{1 + \left(\frac{\sqrt{n}}{n-1}\right) \underbrace{\boxed{\dfrac{(\bar{x}-\mu_0)^2}{s^2/\sqrt{n}}}}_{t}} = \frac{1}{1 + \frac{\sqrt{n}}{n-1}\, t}$$

test is $\lambda \le k \rightsquigarrow$ reject H_0

$$\frac{1}{1 + \frac{\sqrt{n}}{n-1}\, t} \le k$$

$$\frac{1}{k} \le 1 + \frac{\sqrt{n}}{n-1}\, t$$

$$\underset{\dfrac{(\bar{x}-\mu_0)^2}{(s^2/\sqrt{n})}}{\underset{\shortparallel}{t}} \ge \left(\frac{1}{k} - 1\right)\frac{n-1}{\sqrt{n}} = K$$

our hyp. test has the form

if $T \geq K$ reject.

t. dist. w/ $n-1$ degrees of freedom