# Math451_Lecture_Notes

Notes by Bojue Wang

## Lecture_1

**Definition 0.1.** A composition law on a set $S$ is a function $S \times S \to S$. (binary operation).

**Example 0.1.**

$$\begin{aligned} \mathbb{R} \times \mathbb{R} &\to \mathbb{R} \\ (a, b) &\mapsto a + b \\ &\mapsto ab \\ &\mapsto e^a b \end{aligned}$$

$$\begin{aligned} \mathbb{Z} \setminus \{0\} \times \mathbb{Z} \setminus \{0\} &\to \mathbb{Z} \setminus \{0\} \\ (a, b) &\mapsto a^b \end{aligned}$$

**Definition 0.2.** A magma is a set with a law of composition.

In a magma, composition law is typically written as multiplication.

$S \times S \xrightarrow{m} S$, $(a, b) \mapsto m(a, b)$, $a \cdot b = ab \equiv m(a, b)$.

**Definition 0.3.** A magma is associative if $(ab)c = a(bc)$ for all $a, b, c \in S$.

**Definition 0.4.** A semigroup is an associative magma.

**Definition 0.5.** An (unit) identity for a composition law $S \times S \to S$ is an element $e \in S$ such that for all $s \in S$, we have $se = es = s$.

| operator | identity |
|---|---|
| $+$ | $0$ |
| $\cdot$ | $1$ |
| $(M_n(\mathbb{R}), \cdot)$ | $I_n$ |

**Definition 0.6.** A monoid is a semigroup with identity.
$(\mathbb{Z}, \cdot)$,$(\mathbb{Z}_{>0}, \cdot)$

**Definition 0.7.** If $(S, \cdot)$ is a magma with identity and $a \in S$, we say $a$ is invertible if there exists $b \in S$ such that $ab = ba = e$.

**Definition 0.8.** A group is a monoid such that every element is invertible.

$\sim$ 1830, Galois, Abel.

Ubiquitous - groups are everywhere
Rich theory - lots of highly non-trivial facts - theorems.

**Remark 0.1.** Inverses in groups are unique. If $ab = ba = e$, then $b$ is an inverse of $a$ and $ab' = b'a = e$. Then $b'(ab) = (b'a)b$. Since $b'(ab) = b'e = b'$ and $(b'a)b = eb = b$, we have $b' = b$.

**Example 0.2.**

| | $(\mathbb{R}, +)$ | $(\mathbb{R} \setminus \{0\}, \cdot)$ |
|---|---|---|
| Unit | $0$ | $1$ |
| Inverse | $-a$ as the inverse of $a$ | $\frac{1}{a}$ as the inverse of $a$ |
| Verbal perspective | $a \leftrightarrow$ translate by $a$ | $a \leftrightarrow$ stretch or extend by a factor of $a$ |

For a general group $(G, \cdot)$, if $g \in G$, we write $g^{-1}$ for its unique inverse.

**Example 0.3.** General linear group $GL_n(\mathbb{C}) = \{T \in M_n(\mathbb{C}) : \det T \neq 0\}$. The identity and inverse of element $A \in GL_n(\mathbb{C})$ can be expressed readily.

**Example 0.4.** Hours on a clock.
Consider the set $\{\bar{0}, \bar{1}, \ldots, \bar{1}\bar{1}\}$ with operation clock rotations. It forms a group $\mathbb{Z}/12\mathbb{Z}$ or $(\mathbb{Z}, +)$, $C_{12}$.

**Example 0.5.** $S_X = \{f : X \to X | f \text{ is bijective}\}$.
identity$\leftrightarrow$ $id_X$
inverse$\leftrightarrow$ inverse function.

**Exercise 1.** $X = \{1, 2, 3, \}$, $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{smallmatrix}\right)$, $\tau = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right)$. Find $\tau\sigma\tau^{-1}$, $\tau^3$ and $\sigma^3$.