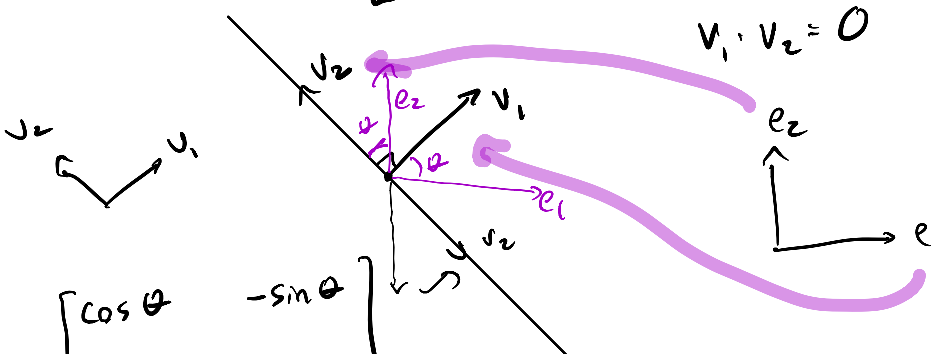
Isometries of $\mathbb{R}^2$

$$O_2(\mathbb{R}) = \{ T \in M_2(\mathbb{R}) \mid T^t T = I_2 \}$$

$$\begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \qquad \begin{aligned} v_1 \cdot v_1 &= 1 \\ v_2 \cdot v_2 &= 1 \\ v_1 \cdot v_2 &= 0 \end{aligned}$$



$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
rotate θ counter clockwise

reflect about x-axis then rotate θ $\longrightarrow$ $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

$$e^{\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$$

Theorem: Classification of symmetries of $\mathbb{R}^2$

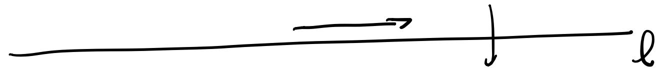
All symmetries of $\mathbb{R}^2$ are either:

orientation preserving $\longrightarrow$
- translations by vector $\vec{v}$
- rotations (about some pt, $\angle\theta$)

- reflections (through a line ℓ)

orientation reversing:

• glide reflections translate along a line, then reflect through it.



Further, we can consider homomorphism

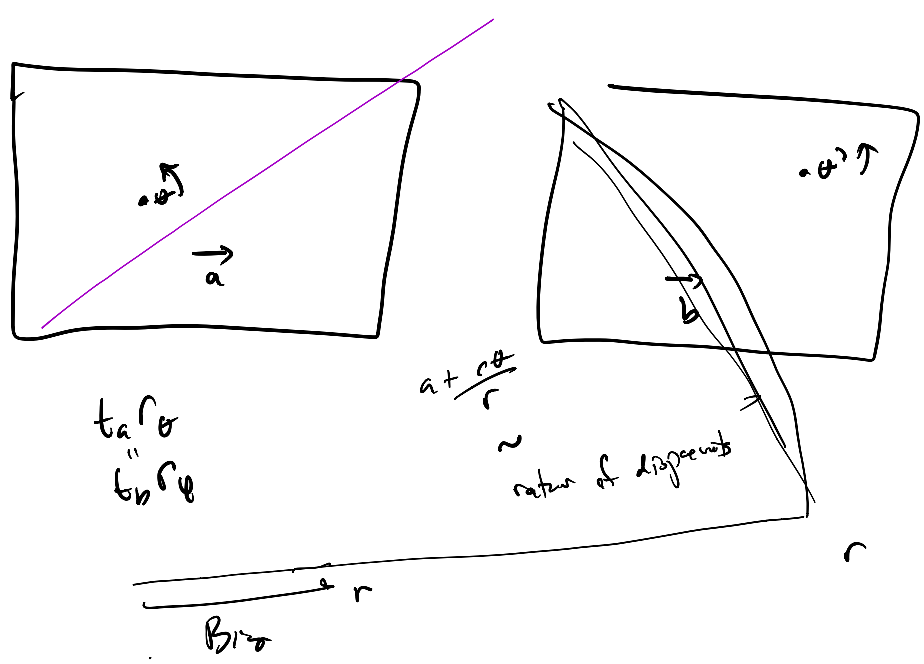
$$\begin{tikzcd} \mathrm{Isom}(\mathbb{R}^2) \arrow[r] \arrow[rr, bend right, "\text{"orientation"}"'] & O_2(\mathbb{R}) \arrow[r, "\det"] & \pm 1 \end{tikzcd}$$

if an isometry maps to 1 it is called "orientation preserving"

— — — — -1 ______ reversing

$$\mathrm{Isom}(\mathbb{R}^2) \longrightarrow O_2(\mathbb{R})$$

$$t_a \cdot T \longmapsto T$$



$t_a r_\theta$
''
$t_b r_\varphi$

$a + \frac{r\theta}{r}$

~ ratio of displacements

$r_{\theta,p}$ = rot θ counter clockwise about p

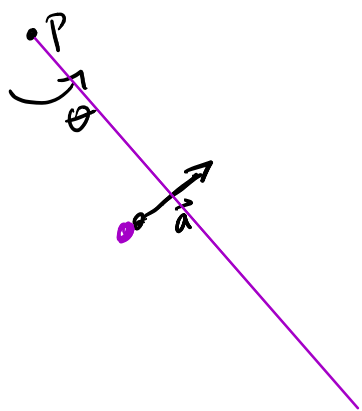
$t_{\vec{a}}$ = tran. $\vec{a}$.

$$t_{\vec{a}} r_{\theta,p} = r_{\theta ?, p?}$$

$$t_{\vec{a}} r_{\theta,p}(q) = q$$

$$t_{\vec{a}}^{-1} t_{\vec{a}} r_{\theta,p}(q) = t_{\vec{a}}^{-1}(q)$$

$$r_{\theta,p}(q) = q - \vec{a}$$



$$t_{\vec{a}} r_{\theta,p} = r_{\varphi,q}$$

Know: every $\mathrm{Isom}(\mathbb{R}^2)$ is of the form

$$t_a T$$

$$T = r_\theta \quad \text{or} \quad r_\theta \tau$$

τ: reflect about x-axis.

now: we've shown

$t_a r_\theta$ = rotation about some point.

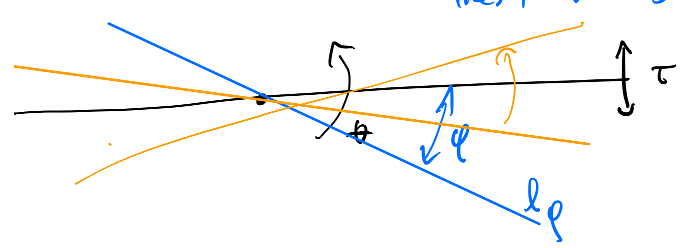
$t_a \boxed{r_\theta}$

- $a = 0 \Rightarrow$ rotation
- $\theta = 0 \Rightarrow$ translation
- $a \neq 0,\ \theta \neq 0 \Rightarrow$ rotation about another pt.

$t_a \boxed{r_\theta \tau} \xrightarrow{a=0} r_\theta \tau$ = reflection:

find a line ℓ preserved by this transformation.

lines through origin are taken to each other by τ & θ.

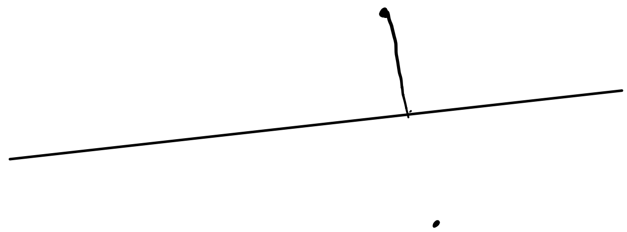


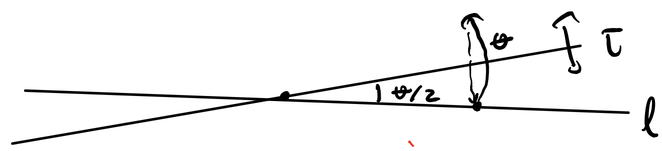
$-\varphi = \varphi + \theta$

$\varphi = -\frac{1}{2}\theta$

$r_\theta \quad \ell_\varphi \to \ell_{\varphi+\theta}$

$\tau \quad \ell_\varphi \to \ell_{-\varphi}$





$\Longrightarrow$ $r_\theta \tau$ = reflection thru ℓ

$t_a r_\theta \tau \quad a \neq 0$ (glide)

$t_a \tau_\ell$