Last time

Symmetries of the plane are of 4 types

orientation-preserving	orientation reversing
• translations	• reflections
• rotations	• glide reflections

Plan:

- Finite subgroups of $O_2(\mathbb{R})$
- Discrete subgroups of $\mathrm{Isom}(\mathbb{R}^2)$ (Wallpaper groups)
- A little about finite subgroups of $O_3(\mathbb{R})$

Finite subgroups of the orth. gp $O_2(\mathbb{R})$

Rotations: if G is a group of symmetries ($G \subset O_2(\mathbb{R})$) then we can consider the subset of $H = \{g \in G \mid g \text{ is a rotation}\}$

then $H < G$

and in fact: H is cyclic generated by

} the smallest counter clockwise rotation in H.

<u>Proof of</u>: We'll show: if H is a group of rotations (about O) such that $\exists$ smallest c.clockwise rotation then H is cyclic, gen by smallest rotation.

let θ be $\measuredangle$ of smallest rotation (radians) $[0, 2\pi)$

and φ any other rotation $\measuredangle$.

$\frac{\varphi}{\theta}$ = some real #, bigger than 1.

$= n + r \qquad r \in [0,1)$

n = integer

$\varphi = n\theta + r\theta \qquad \varphi - n\theta = r\theta$

$H \ni \rho_\varphi \circ \rho_\theta^{-n} = \rho_{\varphi - n\theta} = \rho_{r\theta}$

$\Rightarrow \rho_{r\theta} \in H \quad r\theta < \theta$

contradicts minimality of θ!

unless $r = 0$

$r = 0 \Rightarrow \varphi = n\theta \quad \rho_\varphi = \rho_\theta^n \quad \square$

<u>Note</u>: if l_1 & l_2 lines

r_{l_1}, r_{l_2} reflections through these, then

$r_{l_1} \circ r_{l_2}$ = rotation through $l_1 \cap l_2$

What is $G \subset O_2(\mathbb{R})$ if G is finite?

Case 1

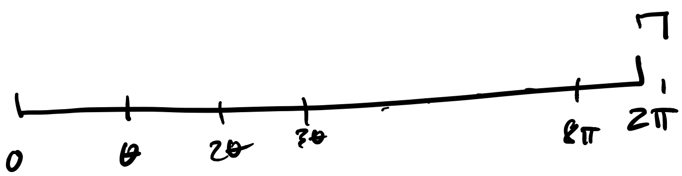
G has no reflections

$\Downarrow$

G is only rotation

$\Downarrow$

$G = \langle \rho_\theta \rangle$ and $\theta = \frac{2\pi}{n}$ some n.



$\rho_{n\theta} = \rho_\theta^n = \rho_\varphi \quad \varphi < \theta$

$\Rightarrow \varphi = 0$

case 2

G has a reflection r (x-axis)

set $H < G$ as before the rotations

then $H = \langle \rho_\theta \rangle$

then every element in G has form ρ_θ^i, or $r\rho_\theta^i$

?

if $g \in G$, either $g \in H$.

or g is a reflection $\Rightarrow rg$ a rotation $\Rightarrow rg \in H$

$g = \rho_\theta^i$ or $rg = \rho_\theta^i$

$g = r^2 g = r\rho_\theta^i$

in second case, describe group structure: $\rho_\theta^i \rho_\theta^j = \rho_\theta^{i+j}$

$r\rho_\theta^i \quad r\rho_\theta^j$

$\| $

$\rho_{i\theta}$

$$\overline{r\rho_\theta r}(v) = \rho_{-\theta}(v)$$

(R, φ)

r both sides

$$r\left(\rho_\theta\left(r(R,\varphi)\right)\right) = r(\rho_\theta(R,-\varphi))$$
$$= r\ (R, -\varphi+\theta)$$
$$= (R, \varphi-\theta)$$

$$\rho_\theta r = r\rho_{-\theta}$$

$$= \rho_{-\theta}(R,\varphi)$$

Def (Abstract dihedral group)

The dihedral gp D_n is the group w/ generators x, y

elements are x^i, $i = 0, \dots, n-1$ $\quad x^n = e$

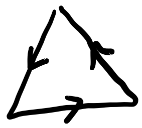
$yx^i, \dots$

such that $y^2 = e$, $\quad xy = yx^{-1}$

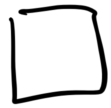
$$x^3 y = xx(xy) = xx(yx^{-1})$$
$$= x(xy)x^{-1}$$
$$= x(yx^{-1})x^{-1}$$
$$= (xy)x^{-1}x^{-1}$$
$$= yx^{-1}x^{-1}x^{-1} = yx^{-3}$$

$$x^i y = yx^{-i}$$

$$(x^2)(yx^3)$$
$$= yx^{-2}x^3 = yx$$



C_3



D_4

C_3

Symmetry groups in $\mathrm{Isom}(\mathbb{R}^2)$

If $G \subset \mathrm{Isom}(\mathbb{R}^2)$ is a finite gp of symmetries
then G fixes a point
(and so, after changing coords, $G \subset O_2(\mathbb{R})$)

why?

centroid of pts
= center of mass if each pt has unit mass.

<u>Define</u> $G \subset \mathrm{Isom}(\mathbb{R}^2)$ is "discrete"
if $\exists$ smallest rotation angle
and $\exists$ smallest translation distance

<u>Theorem</u> G has one of 17 types . . .

"wallpaper groups"

Given such a G, can map G to $O_2(\mathbb{R})$

i.e. for $g = t_a \cdot T$, map g to T

gives a homomorphism

$$G \xrightarrow{f} O_2(\mathbb{R})$$

let $L = \ker(f) = \{ t_a \mid t_a \in G \}$

"Lattice"

think about elements in L as vectors $t_{\vec{a}} \longleftrightarrow \vec{a}$ in $\mathbb{R}^2$

know $\exists\ \vec{a} \in \mathbb{R}^2$ in L of minimal length.

$$\underset{\vec{a}}{\longrightarrow}$$

<u>Definition</u> the image of $f = \{ T \in O_2(\mathbb{R}) \mid t_a T \in G \text{ some } t_a \}$

is called the "point group" of G

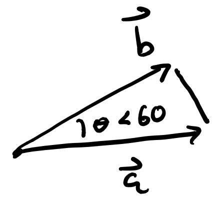
denoted $\overline{G}$

<u>turns out</u>: (after HW)

that if $\overline{g} \in \overline{G}$, $l \in L$ then $\overline{g}\, l \in L$.

__Claim:__ $\bar{G}$ cannot contain rotations of less than 60°.

__Pf:__



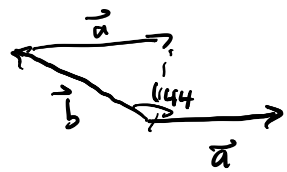
$\vec{a} - \vec{b} \in L$

smaller

$\Rightarrow \measuredangle$ at least 60°.

$\Rightarrow$ rotation has order at most 6

also 72° not allowed for rotations.



$|\vec{b} + \vec{a}| < |\vec{a}|$

~~x~~.

only allowable $\measuredangle$'s are $\frac{2\pi}{n}$, $n = 2, 3, 4, 6$.