Symmetries of $\mathbb{R}^n$

$\mathrm{Isom}(\mathbb{R}^n)$

$\mathbb{R}^n < \mathrm{Isom}(\mathbb{R}^n)$

$v \longleftrightarrow t_v$

$\mathrm{Isom}(\mathbb{R}^n) \xrightarrow{\varphi} O_n(\mathbb{R})$

surjective

1^{st} isom thm

$\Downarrow$

$\mathrm{Isom}(\mathbb{R}^n) / \ker\varphi \simeq O_n(\mathbb{R})$

$$\frac{\mathrm{Isom}(\mathbb{R}^n)}{\mathbb{R}^n} \simeq O_n(\mathbb{R})$$

$O_n(\mathbb{R}) \xrightarrow{\det} \pm 1$

$\ker(\det)$ "orientation preserving"

$\parallel$

$SO_n(\mathbb{R})$

$\det = -1$

coset of $SO_n(\mathbb{R})$

$n = 2$ $\quad$ $SO_2(\mathbb{R})$ rotations

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

orientation reversing : reflections

$n = 3$ $\qquad SO_3(\mathbb{R})$ $\qquad$ rotations

reflection twice

$T \in M_n(\mathbb{R})$ $\quad n$ odd then T has an eigenvector.

$$\chi_T(x) = \det(xI_n - T) \quad \text{degree } n.$$

$$x^n + \cdots$$

has a root λ

$$(\lambda I - T)v = 0$$

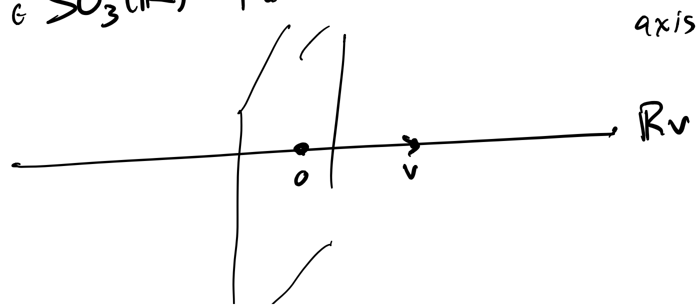
singular $\Downarrow$

v an eigen vector.

if $T \in O_3(\mathbb{R})$ $\qquad Tv = \lambda v \implies \lambda = \pm 1$

$$\|v\| = \|Tv\| = \|\lambda v\| = |\lambda|\,\|v\| \implies |\lambda| = 1 \quad \Uparrow$$

if $T \in SO_3(\mathbb{R})$ then T is a rotation through some axis



$\checkmark$

So T preserves plane $P \subset \mathbb{R}^3$ perp to v.

i.e. T gives an isometry $T|_P$ of P

$\Rightarrow T|_P$ either rotation or reflection

T as a matrix has block-diagonal form

$$T = \left[\begin{array}{c|c} T|_P & 0 \\ \hline 0 & \pm 1 \end{array}\right]$$

basis $\underbrace{b_1, b_2}_{\text{basis for } P}$, v ← e.vector

if det = 1 either $T|_P$ and ± 1 have det 1

or $T|_P$ & ± 1 have det -1

$\Rightarrow T|_P$ is a rotation & $\ell = \mathbb{R}v$ is fixed

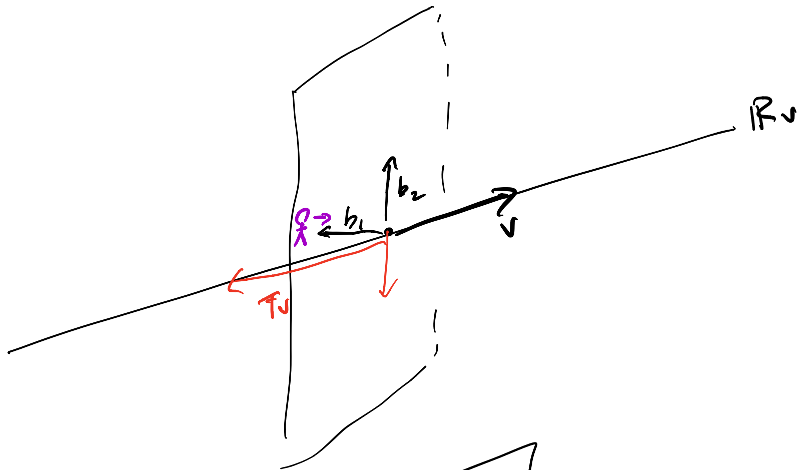
or $T|_P$ is a reflection & -1 in corner

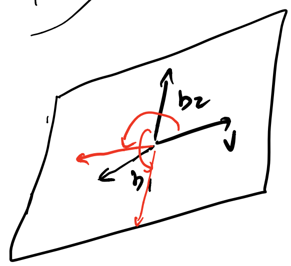
can find b_1, b_2 an orth basis for P w/

$T b_1 = b_1$
$T b_2 = -b_2$

$$\Rightarrow T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

180° degree rotation in $\langle b_2, v \rangle$ plane.





Def $\mathbb{H} = \{a+bi+cj+dk \mid a,b,c,d \in \mathbb{R}\}$

Quaternions

$i^2 = j^2 = k^2 = -1$

$ij = k = -ji$

$q = a+bi+cj+dk \qquad \bar{q} = a-bi-cj-dk$

$\|q\| = q\bar{q} = a^2+b^2+c^2+d^2 \in \mathbb{R}$

$\mathbb{H}^\times = \{q \in \mathbb{H} \mid q \neq 0\}$

group under mult. $\qquad q \cdot \dfrac{\bar{q}}{\|q\|} = 1$

$q^{-1} = \dfrac{\bar{q}}{\|q\|}$

$\mathbb{R}^3 \longleftrightarrow$ "pure quaternions"

$= \{bi + cj + dk \mid b, c, d \in \mathbb{R}\} \subset \mathbb{H}$

and then if $v \in \mathbb{R}^3 \subset \mathbb{H}$ and $q \in \mathbb{H}^*$

map $v \longmapsto qvq^{-1}$ describes a rotation in $\mathbb{R}^3$

get a map $\mathbb{H}^\times \xrightarrow{\varphi} SO_3(\mathbb{R})$

$q \longmapsto (v \mapsto qvq^{-1})$

if $q \in \mathbb{R}^\times \subset \mathbb{H}^\times$ then $q \in \ker \varphi$

$$\mathrm{Spin}_3(\mathbb{R}) = \mathbb{H}^\times / \mathbb{R}^\times \longrightarrow SO_3(\mathbb{R})$$

$$\tilde{R}_1 \longrightarrow R$$

$$\tilde{R}_2 \nearrow$$

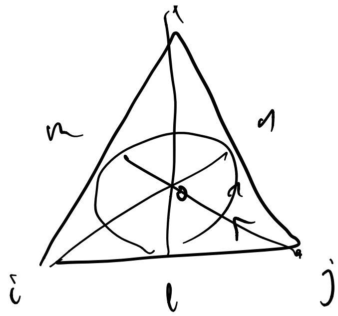
$qvq^{-1} = vqq^{-1} = v$

if $q = \lambda \in \mathbb{R}^\times \subset \mathbb{H}^\times$

$q \cdot p = p \cdot q$

$\mathbb{O} = \{a + bi + cj + dk + el + fm + gn + ho\}$

$i^2 = j^2 = \cdots = -1$



"Fano plane"

Theorem (Frobenius 1880's)

if A is a division algebra of finite dimension over $\mathbb{R}$

then $A = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

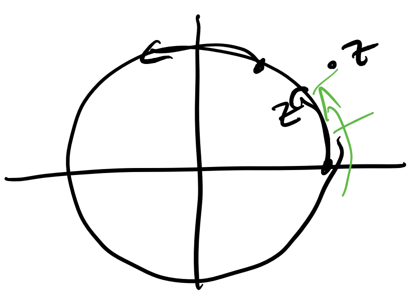
not associative

Alternate approach to rotations in higher dim

Clifford algebras

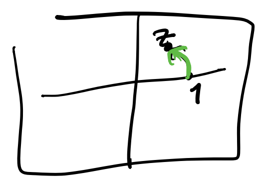
only reasonable ways to describe Spin phenomena in higher dim.

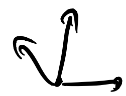
$wz = w \Rightarrow z = 1$



choose $z \neq 1$ in $\mathbb{C}$

$\|z\| = 1$





<u>Thm</u> 1960 Adams

Tangent bundles for spheres are only parallelizable in dims 1, 3, 7.