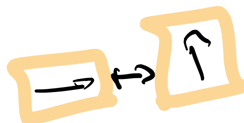
$R: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

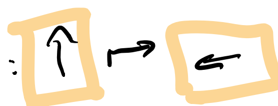
$R: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix}$$

x^2+1

$R:$ 

$e_1 \mapsto e_2$

$R:$ 

$e_2 \mapsto -e_1$

$Rv = iv \qquad (R - iI)v = 0$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \to \begin{bmatrix} 1 & -i \\ 1 & -i \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} -1 \\ i \end{bmatrix}$$

$\lambda \in \mathbb{C} \qquad r \in \mathbb{R}^\times$

$\lambda r = s$

$\lambda r \in \mathbb{R} \Rightarrow \lambda \in \mathbb{R}$

$\lambda = s r^{-1}$

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad v \in \mathbb{R}^2$

$\lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in $\mathbb{R}^2$

$$\underbrace{Tv}_{\text{in } \mathbb{R}^2} = \underbrace{\lambda v}_{\text{in } \mathbb{R}^2}$$

if $\lambda \in \mathbb{R}$

$T - \lambda I$

λv_1

λv_2

Groups are verbs

Theorem (Cayley's theorem)
every finite group is isomorphic to a subgroup of a permutation group S_n.

i.e. $\exists$ an $\underbrace{\text{injective homomorphism } G \to S_n}_{\text{"permutation representation of } G\text{"}}$ for some n.

Proof: Consider the action of G on itself by left multiplication

$$G \times G \longrightarrow G$$

$$\varphi : G \longrightarrow S_G$$
$$g \longmapsto \varphi(g) \qquad \varphi(g)(h) = gh$$

know: $\operatorname{im} \varphi \cong G/\ker\varphi = G$ $\boxed{\ker\varphi = (e)}$

if $\varphi(g) = id_G$

$\varphi(g)(e) = g = id_G(e)$
$\Rightarrow g = e$

$\square$

this is called the "regular representation"

Excursion to permutation groups

As we've seen: can represent elements of S_n as diagrams:
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

A cycle is a permutation of the form

$1 \to 2$
$2 \to 3$
$3 \to 1$

given any permutation $\sigma \in S_n$, then can consider orbits of $\langle \sigma \rangle$

$$\{1, \sigma(1), \sigma^2(1), \sigma^3(1) \dots, \sigma^l(1)\}$$

$$\sigma^{l+1}(1) = 1 \quad \left\{ \begin{array}{l} \text{if } \sigma^{l+1}(1) = \sigma^i(1) \\ \sigma^{-i} \circ \sigma^{l+1} \cdot 1 = \sigma^{-i} \sigma^i \cdot 1 \\ \sigma^{l-i+1}(1) = 1 \\ \Rightarrow \text{first repeat takes us to } 1 \end{array} \right.$$

in other words: orbits under $\langle \sigma \rangle$ are cycles

Know: $\{1, \dots, n\}$ is a disjoint union of orbits of $\langle \sigma \rangle$

and on each orbit, action of σ is described by a cycle.

$$i \mapsto \sigma(i) \mapsto \sigma^2 \cdots \sigma^{l}(i)$$

notation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \longleftrightarrow (1432)$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \longleftrightarrow (12)(3)(4)$$

$$= (12)$$

<u>Theorem</u> every permutation $\sigma \in S_n$ can be written as a product of disjoint cycles, and this is unique up to reordering.

<u>Lemma</u>: if cycles are disjoint they commute.

$$(123)(45) = (45)(123)$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 4 & 3 \end{pmatrix}$$

$$(1\,2)(3\,6)(4\,5)$$

$(i\,j)$ "transposition"

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1\,2\,3)$$

$$= (1\,3)(1\,2)$$

$$(1\,2\,3\,4) = (i\,j)(k\,l)\cdots(\ \ )?$$

$$\boxed{\begin{matrix} (1\,4)(1\,3)(1\,2) \\ (1\,2)(2\,3)(3\,4) \end{matrix}}$$

Observation: Every cycle is a product of transpositions

$$(1\,2 \cdots n) = (1\,n)(1\,(n-1))\cdots(1\,3)(1\,2)$$

$\Rightarrow$ every permutation is a product of transpositions.

$$e = (1\,2)(1\,2)$$

Def A permutation is <u>even</u> if it can be written as an even # of transpositions, <u>odd</u> - - - - - - odd - - - - -.

Lem: Permutation are even xor odd.

Pf. consider action of permutations in S_n on polynomials in n variables $x_1, \dots, x_n$

consider $d(x_1, \dots, x_n) = \prod_{i<j} (x_i - x_j)$

$(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$

exercise: if $\sigma = (i\ j)$ then $\sigma(d) = -d$

$\Rightarrow \sigma$ arbitrary, product of k transpositions

$\sigma(d) = (-1)^k d$

Alt. proof

S_n acts on $\mathbb{R}^n$ as linear transformations by permuting basis $e_1, \dots, e_n$

transposition: after change of basis

$$\begin{bmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

$\sigma \rightsquigarrow T_\sigma$ if σ a transposition
$\det \sigma = -1$

$$T_{\sigma\tau} = T_\sigma T_\tau$$

if σ = prod $\tau_1, \dots, \tau_k$ transp.

then $\det T_\sigma = \prod \det T_{\tau_i} = (-1)^k$

<u>Def</u> $A_n = \{\sigma \in S_n \mid \sigma \text{ even}\} = \ker(\mathrm{sgn})$

$$S_n \xrightarrow{\mathrm{sgn}} \{\pm 1\}$$
$$\sigma \longmapsto \det(T_\sigma)$$
$$\sigma\tau \longmapsto \det(T_{\sigma\tau}) = \det(T_\sigma T_\tau)$$
$$= \det(T_\sigma)\det(T_\tau)$$

<u>Fun fact:</u> A_5 = orientation preserving isometries of the icosahedron.

A_n, $n \geq 5$ are <u>simple</u> groups (no nontrivial normal subgroups)

Given $\sigma = c_1 c_2 \cdots c_k$ permutation in S_n
c_i disjoint cycles.

note $\sigma^n = (c_1 c_2 c_3 \cdots c_k)^n = c_1^n c_2^n \cdots c_k^n = e$

iff $c_i^n = e$ all i.

$\text{ordr}(\sigma) = \text{lcm}(\text{ordr } c_i)$ each i.

S_5 largest ordr

$2+3$

S_6

$1+5$ $\quad$ $1+3+2$

$\cancel{3+3}$ $\quad$ $4+2$

S_{10}

$2+3+5$

$G \circlearrowright G$

ghg^{-1}