Conjugaton

we say $g_1, g_2 \in G$ are conjugate if

$$g_1 = h g_2 h^{-1} \text{ some } h \in G$$

G acts on itself via "conjugaton"

$$g \cdot h = g h g^{-1}$$

$e \cdot h = h$

$g_1 \cdot (g_2 \cdot h) = (g_1 g_2) \cdot h$

$e h e^{-1} = h$

$g_1 \cdot (g_2 h g_2^{-1})$

$= g_1 (g_2 h g_2^{-1}) g_1^{-1}$

$= (g_1 g_2) h (g_1 g_2)^{-1}$

$= (g_1 g_2) \cdot h$

g_1 conj to g_2

$\iff$ same orbit w/r/to this conj. action

$\Rightarrow$ equiv. relation.

Notation: $g_1 \sim g_2$ conjugates fr.

Ex: S_n

$$(15)(1234)(15)^{-1}$$

$$= (1)(2345) = (5234)$$

$$(541)(1234)(145) = (1523)$$

$$= \cancel{(4235)}$$

$= (5\ 2\ 3\ 1)$

Moral: if G acts on a set X and $h, g \in G$, then action of hgh^{-1} looks like the action of g after X is relabelled according to h.

$g(x) = y \Rightarrow$

$$hgh^{-1}(hx) = hy$$

$$\shortparallel \qquad\qquad \shortparallel$$

$$hg(x) = h(y)$$

note if G is Abelian

$$ghg^{-1} = h$$

Def The center of a group G, $Z(G)$

$$= \{ g \in G \mid gh = hg \text{ all } h \in G \}$$

note: if $z \in Z(G)$ then conjugacy class of z is just z

$$gzg^{-1} = z \text{ all } g.$$

also: $z \cdot g = g$ conj action by z is trivial.

in fact: $z \cdot g = g$ all $g \iff z \in Z(G)$

$$z \cdot g = g \iff zgz^{-1} = g \iff zg = gz \text{ all } g$$

$$\iff z \in Z(G)$$

$\Rightarrow Z(G) =$ kernel of conj action.

$\Rightarrow Z(G) \triangleleft G$.

<u>Goal</u>: take group, break it up into "simpler pieces"

if $\exists\, N \triangleleft G$ then G is "broken up" into G/N & N.

"devissage"

<u>Def</u> A subgroup $N < G$ is called <u>characteristic</u> if $\forall\, \varphi: G \to G$ automorphisms, $\varphi(N) = N$.

<u>ex</u>: $Z(G)$ char G

$$\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$$

<u>Note</u>: N chr $G \Rightarrow N \triangleleft G$

because if $g \in G$ then $G \to G$, $h \mapsto ghg^{-1}$ is an aut.

<u>Thursday's plan:</u>

towards a partial converse to Lagrange's thm.

$|H| \mid |G|$ if $m \mid |G| \overset{?}{\Rightarrow} \exists H < G$ w/ $|H| = m$

<u>yes</u> if m is a <u>prime power</u>!

<u>The Class equation</u> = path to the holy grail

keys to the kingdom

$$|G| = \sum_{\substack{\text{conj} \\ \text{classes } C}} |C| = \underbrace{\sum_{\substack{\text{conj} \\ \text{classes } C \\ \text{of size } 1}} 1}_{|Z(G)|} + \sum_{\substack{\text{nontriv.} \\ \text{conj} \\ \text{classes } C}} |C|$$

The Icosahedral Rotation group

(smallest nonabelian simple group)

Count elements:

acts on faces of icosahedron

$$|\text{orbit}| = \frac{|\text{grp}| = ?}{|\text{stab}| = 3}$$

$$\underset{20}{\shortparallel}$$

$$\Rightarrow \underline{|G| = 60}$$

Next: we'll show $G \simeq A_5$

$$G \longrightarrow S_5$$

G acts on 5 tetrahedra by permuting them.

consider the image: we'll construct 60 things in image
all be in A_5

observe: get 3-cycles (all of them)

look at 2 other ~~tetra~~hedra, rotate about a vertex in one lying above face of other.

how many 3 cycles $\binom{5}{3} \cdot 2 = 20$

$$(123)(345) = (12345)$$

$\Rightarrow$ also have all 5 cycles.

$1 \cdot 4! = 24$ so far

$20 + 24 + 1$ elmts.

id

$G \longrightarrow S_5$ at least 45 elmts in image in S_5

$G/\ker \simeq \text{im}$

$\Rightarrow |\text{im}|\,|\ker| = |G|$

$|G|/|\text{im}| = |\ker|$

$\Rightarrow |\text{im}| \,\big|\, |G| = 60 \quad |\text{im}| \geq 45$

$\Rightarrow |\text{im}| = |G|$

$\Rightarrow$ map is injective $\Rightarrow \ker = e.$

notice: 3 cycles & 5 cycles are <u>even</u>.

$(1\,2\,3)$ $\quad$ $(1\,2\,3\,4\,5)$

$= (1\,3)(1\,2)$ $\quad$ $= (1\,5)(1\,4)(1\,3)(1\,2)$

consider the subgroup of S_5 gen. by 3-cycles & 5-cycles.

H

know $H \subset \text{im}$ 60

$\cap$

A_5 even permutations

order 60

$A_5 = \ker(S_5 \to \pm 1)$

$|S_5| = 5! = 120$

$\frac{S_5}{A_5} \simeq \pm 1$

$45 \leq |H| \,\big|\, |A_5| = 60$

$\Rightarrow |H| = 60 \qquad H \subset A_5 \Rightarrow \underset{\mathrm{im}}{\underset{\shortparallel}{H}} = A_5$

$G/\underset{e}{\underset{\shortparallel}{\ker}} \cong \mathrm{im} = A_5 \qquad \boxed{G \cong A_5}$

Conjugacy classes

G acts on faces − 20
edges − 30
vertices − 12

conj classes:

rotation about a face by $120°$

change of basis idea $\Rightarrow$ rot. about any other face by $120°$ is conj to any other.

$\rightarrow g =$ rot 120 about F

and if $hF = F'$

then $hgh^{-1} =$ rot 120 about F'

$|$conj class of $g| = 20$? accounts for all rots about faces of $\pm 120°$

$$\underset{e}{1} + \underset{\text{rot }\Delta}{20} + \underset{\text{flip edges}}{15} + \underset{\text{small rot on corner}}{12} + \underset{\text{big rot on corner}}{12}$$