From last time:

G = rotational (orientation preserving) isometries of the icosahedron / dodecahedron

First counted $|G| = 60$

G acts on 5 tetrahedra ← set whose elmts are

$G \xrightarrow{\varphi} S_5$

showed that all 3-cycles & 5-cycles are in image of φ

3-cycles & 5-cycles generate $A_5 \subset S_5$

$\Rightarrow A_5 \subset \operatorname{im}\varphi$

$|A_5| = 60 \qquad 60 \,\big|\, |\operatorname{im}\varphi| \,\big|\, |G| = 60$

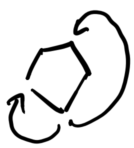
$\Rightarrow \varphi$ is injective

and $\operatorname{im}\varphi = A_5$, $\operatorname{im}\varphi \cong G$

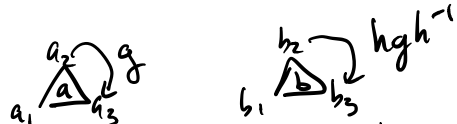
Conjugacy classes

enumerate all elements of G

- identity (1)
- triangular faces, rotate 120° around one (20)
- vertices - 5 fold symmetry (12 big) (12 small)
- edges - 180° rot. 1 for each pair of edges (15)



if $g(x) = y$ then $hgh^{-1}(hx) = hy$



choose $h \in G$ s.t. $h\left(\underset{a_1\ \ a_3}{\overset{a_2}{\triangle_a}}\right) = \underset{b_1\ \ b_3}{\overset{b_2}{\triangle_b}}$

$h(a_2) = b_2$

$g(a_1) = a_2 \rightsquigarrow hgh^{-1}(\underset{\substack{\shortparallel \\ b_1}}{ha_1}) = \underset{\substack{\shortparallel \\ b_2}}{ha_2}$

Final observation to show G simple.

<u>notice</u>: if $N \trianglelefteq G$, $n \in N$, $n' \sim n$

$\Rightarrow n' \in N$ def of normal.

$(n' \sim n \iff gng^{-1} = n'$ for some $g)$

$\Rightarrow$ if $n \in N$ then the full conj. class of n is in N.

$\Rightarrow N$ is a union of conj classes (including identity)

"class equation"

$60 = 1 + 12 + 12 + 20 + 15$

$|N|$ = sum of some of these including 1.

$|N| \,\big|\, |G|$. $\Rightarrow |N| = 1$ or 60.

Towards Sylow Theorems

if $p^n \mid |G|$ p prime $\Rightarrow \exists\, H < G$ $|H| = p^n$

Important role is therefore played by "p-groups"

Def A p-group is a group G, $|G| = p^n$ some n.

Digression: p-groups:

Class equation:

$$|G| = \underbrace{|Z(G)|}_{\text{conj classes of size 1}} + \underbrace{\sum |C|}_{C \text{ conj class of size} > 1}$$

G acts by conj on itself these are the orbits

Lem: if G is a p-group, C a conj class

$\Rightarrow$ either $|C| = 1$

or $p \mid |C|$

$\Rightarrow p \mid |G|$, $p \mid |C|$

$\Rightarrow p \mid \sum |C|$

$\Rightarrow p \mid |G| - \sum |C| \Rightarrow p \mid |Z(G)|$

$\Rightarrow$ $\boxed{Z(G) \neq e \quad \text{if } G \text{ is a } p\text{-group}}$

if G is a p-group $\quad Z(G) \neq (e)$, $\quad Z(G) \trianglelefteq G$

$\Rightarrow$ can form $G/Z(G)$ another p-group smaller.

$$Z\left(G/Z(G)\right) \neq e$$

"the ascending central series of G"

if G has a acs which "exhausts" the gp.

we say G is "nilpotent"

Application:

if $|G| = p \Rightarrow |Z(G)| = p$

$\Rightarrow G$ is Abelian.

if $g \in G$, $g \neq e$ then $\langle g \rangle \neq e$

$|\langle g \rangle| \mid |G|$

$\Rightarrow \langle g \rangle = G \qquad G = C_p$

$|G| = p^2 \qquad |Z(G)| = p, p^2$

if $|Z(G)| = p$

know $Z(G) = \langle z \rangle = \{e, z, z^2, \dots, z^{p-1}\}$

G union of cosets $Z(G) \cup gZ(G) \cup g^2 Z(G)$
$\vdots$

$|G/Z(G)| = p$ cyclic

if $gZ(G)$ nontrivial coset

$\Rightarrow G = \bigcup_i g^i Z(G) = \{g^i z^j\}$

$Z(G) = \{e, z, z^2 \dots\}$

$(g^i z^j)(g^k z^l) = g^i g^k z^j z^l$

$= g^{i+k} z^{j+l}$

$= g^k g^i z^l z^j$

$= g^k z^l g^i z^j$

$$= (g^k z^\ell)(g^i z^\ell)$$

$$\Rightarrow \left(|G| = p^2 \Rightarrow G \text{ Abelian.} \right)$$

$|Q_8| = 8$ $\quad Q_8 = \{\pm 1, \pm i, \pm j \pm k\}$

$$Z(Q_8) = \{\pm 1\}$$

$$Q_8 / Z(Q_8) = V_4$$

"The Extension problem"

Given $N, \overline{G}$ groups can we classify all groups G containg N as a normal subgroup such that $G/N \cong \overline{G}$

Answer is usually horrible, but occasionally beautiful.

<u>Easiest possible way:</u>

"Product"

Def if G_1, G_2 groups

$$G_1 \times G_2 = \{(g_1, g_2) \mid g_i \in G_i\}$$

$$(g_1, g_2)(g_1', g_2') = (g_1 g_1', g_2 g_2')$$

Can show: if given $N, \overline{G}$ then

$G \equiv N \times \overline{G}$ contains $\{(n, e) \mid n \in N\} \lhd G$

$\cong$ to N

an $G/$that $\cong \overline{G}$