Recall:

The class equation: expresses the fact that G is a disjoint union of orbits under the "conjugation action"

G acts on itself via

$g, x \in G \qquad g \cdot x = gxg^{-1}$

orbits $\longleftrightarrow$ "conjugacy classes" $\qquad x \sim y$

$x = gyg^{-1}$ some g.

$$|G| = \sum_{C \text{ conj classes}} |C| = \underbrace{\boxed{\sum_{\substack{C \text{ conj classes} \\ |C|=1}} 1}}_{= |Z(G)|} + \sum_{\substack{C \text{ conj classes} \\ \text{order} \geq 2}} |C|$$

$x \in Z(G) \iff$ conj class of $x = \{x\}$ $\quad gxg^{-1} = x$ all $g \in G$

If C conj class then C is an orbit under conj action.

if $C = [x] = \{gxg^{-1} \mid g \in G\}$

then $|C| = \dfrac{|G|}{|\mathrm{Stab}_G(x)|} = [G : C_G(x)]$

$\mathrm{Stab}_G(x) = \{g \in G \mid gxg^{-1} = x\} = \{g \in G \mid gx = xg\}$

$= C_G(x) < G$

$C = [x]$ $\quad |C|$ $\quad$ C an orbit of size > 1

Alt class equ:

$$\boxed{|G| = |Z(G)| + \sum_{\substack{\text{some} \\ x \in G \\ \text{(one from each} \\ \text{nontriv. conj. class)}}} [G : C_G(x)]}$$

Sylow Warmup:

(Sylow ⇒ if $p^n \mid |G|$ ⇒ $\exists H < G$, $|H| = p^n$ $\quad$ p prime.)

Thm (Cauchy)

If p prime, $p \mid |G|$

then $\exists g \in G$ s.t. $\operatorname{ord}(g) = p$

(and so $\langle g \rangle < G$ w/ $|\langle g \rangle| = p$)

Pf: Induction on $|G|$

base case: $|G| = p$

if $g \in G$, $g \neq e$ then $o(g) > 1$ some $g \neq e$

and $o(g) \mid |G| = p$ ⇒ $o(g) = p$ ✓

Induction: write $|G| = |Z(G)| + \sum_{\text{some } x\text{'s}} [G : C_G(x)]$

if $p \mid |C_G(x)|$ $\qquad$ note: $|C_G(x)| \lneq |G|$

then by induction $\exists\, g \in C_G(x)$ $ord(g) = p$

done.

so WLOG $p \nmid |C_G(x)|$ all x's as above.

$$p \mid |G| = |C_G(x)| \cdot [G : C_G(x)]$$

$$\Rightarrow p \mid [G : C_G(x)] \text{ all } x\text{'s.}$$

$$p \mid |G|,\ p \mid [G : C_G(x)] \Rightarrow p \mid \underbrace{|G| - \sum [G : C_G(x)]}_{|Z(G)|}$$

$\Rightarrow p \mid |Z(G)|$ and

WLOG, G is Abelian.

Choose $g \in G$ (Abelian) $g \neq e$

if $p \mid o(g)$ say $o(g) = p\ell$ then

$o(g^\ell) = p$

so use g^ℓ ✓

if $p \nmid o(g)$ consider $G/\langle g \rangle$

order is smaller, $p \nmid |\langle g \rangle| = o(g)$

$$\Rightarrow p \mid |G/\langle g \rangle|$$

$$|G/\langle g \rangle|\,|\langle g \rangle| = |G|$$

by induction, $\exists\, h \in G/\langle g \rangle$ s.t. $o(h) = p$

$h = g' \langle g \rangle$ $\quad h^p = e$ in $G/\langle g \rangle$

$g' \neq e$

$(g' \langle g \rangle)^p = (g')^p \langle g \rangle$

$= \langle g \rangle$

$\Rightarrow g'^p \in \langle g \rangle$

$g' \notin \langle g \rangle$

$\Rightarrow p \mid o(g')$

To finish need to show:

if $H \triangleleft G$, $g' \in G \setminus H$, $(g')^p \in H$

$\Rightarrow p \mid o(g')$ $\quad$ (i.e. if $(g')^n = e$ then $p \mid n$)

Hint: if $(g')^n = e$ then $(g')^n$ is in H

$\square$

<u>Example</u> if $|G|=20$ then $\exists N \triangleleft G$ $|N|=5$

<u>why?</u> $5 \mid 20$ 5 is prime $\Rightarrow \exists N < G$ $|N|=5$
Cauchy

$$G \circlearrowright G/N = \{ gN \mid g \in G \}$$

via $g \cdot g'N \equiv gg'N$

how many cosets are there? 4

$g_1H \xrightarrow{g_2 g_1^{-1}} g_2H$

$$G \xrightarrow{\varphi} S_4 = S_{G/N} \qquad |S_4| = 4! = 24$$

$|20|$ $|24|$

G 20: g_0H, g_1H, g_2H, g_3H

$$G/\ker\varphi \simeq \operatorname{im}\varphi \qquad |\ker\varphi| = 5, 10, 20$$

$$|\operatorname{im}\varphi| \mid 24$$

also $\ker\varphi \subset \operatorname{Stab}_G N$ $\qquad |\operatorname{Stab}_G N| = \frac{|G|}{|\operatorname{orb} H|} = 5$

$$|\ker\varphi| \mid 5 \quad \Rightarrow |\ker\varphi| = 5$$

$\Rightarrow \ker\varphi \triangleleft G$ has order 5.

$\underset{\text{order } 5}{\ker\varphi} < \underset{\text{order } 5}{\operatorname{Stab}_G N}$ $\qquad \underset{\text{order } 5}{N} < \operatorname{Stab}_G N$

$$\Rightarrow N = \operatorname{Stab}_G N = \ker\varphi \triangleleft G. \quad \square$$

If $|G| = 2m$ and $H < G$ $\quad |H| = m$

then $H \lhd G$.

<u>Pf:</u> $G \circlearrowright G/H$ left mult H.

$G \xrightarrow{\varphi} S_2$ action is nontrivial (since one orbit)

$\Rightarrow \varphi$ onto.

$\Rightarrow \ker\varphi \lhd G$ w/ $|\ker\varphi| = m$

$\underset{m}{H} < \underset{m}{\mathrm{Stab}_G H}$ $\qquad |\mathrm{Stab}_G H| = \dfrac{|G|}{2} = m$

$H = \underset{m}{\mathrm{Stab}_G H} \underset{=}{\supset} \underset{m}{\ker\varphi} \lhd G$ $\quad \square$

Aside:

Putting groups back together from their parts.

Recall: if G_1, G_2 groups, can form

$$G_1 \times G_2 = \{(g_1, g_2) \mid g_i \in G_i\}$$

external direct product

w/ operation $(g_1, g_2) \cdot (g_1', g_2') = (g_1 g_1', g_2 g_2')$

Given G, how do we tell if $G \cong G_1 \times G_2$ some G_i's?

"internal direct product"

Def G is an internal direct product of subgroups N_1, N_2 if $N_1, N_2 \triangleleft G$, $N_1 \cap N_2 = \{e\}$ and $N_1 N_2 = \{n_1 n_2 \mid n_1 \in N_1, n_2 \in N_2\} = G$

Notation $G = N_1 \dot{\times} N_2$

Thm If $G = N_1 \dot{\times} N_2$ then $G \cong N_1 \times N_2$

Proof Define $\varphi : N_1 \times N_2 \longrightarrow G$

$$\varphi(n_1, n_2) = n_1 n_2$$

this is a homomorphism!

$$\varphi((n_1, n_2)(n_1', n_2')) \stackrel{?}{=} \varphi(n_1, n_2)\, \varphi(n_1', n_2')$$

$$\qquad // \qquad\qquad\qquad\qquad \| $$

$$\qquad\qquad\qquad\qquad\qquad n_1 n_2 n_1' n_2'$$

$$\varphi(n_1 n_1', n_2 n_2') = n_1 n_1' n_2 n_2'$$

$$n_1' n_2 \stackrel{?}{=} n_2 n_1' \checkmark$$

$$n_1' n_2 (n_1')^{-1} n_2^{-1} \in N_1 \cap N_2 = \{e\}$$

$$N_2 \ni \Big(\underbrace{n_1' n_2 (n_1')^{-1}}_{\in n_1' N_2 (n_1')^{-1} = N_2}\Big) n_2^{-1} \quad \text{in } N_2 \quad \text{in } N_2$$

$$= n_1' \Big(\underbrace{n_2 (n_1')^{-1} n_2^{-1}}_{\in N_1}\Big) \in N_1, \quad n_1' \in N_1$$

$$\Rightarrow n_1' n_2 (n_1')^{-1} n_2^{-1} = e$$

$$n_1' n_2 (n_1')^{-1} = n_2$$

$$\boxed{n_1' n_2 = n_2 n_1'}$$

φ surj. since $G = N_1 N_2$

φ injective? $\ker\varphi = \{(n_1, n_2) \mid n_1 n_2 = e\}$

$$n_1 n_2 = e \Rightarrow \underset{N_2}{n_2} = \underset{N_1}{n_1^{-1}}$$

$$\Rightarrow n_1^{-1} = n_2 \in N_1 \cap N_2 = \{e\}$$

$$\Rightarrow n_1^{-1} = e = n_2, \quad n_1 = e$$

so injective $\checkmark$

Tricks later will show:

if $|G| = 15$ then $\exists N_5, N_3 \triangleleft G$

order $5, 3$

$3, 5 \mid N_5 N_3 < G$

$N_3 \cap N_5 < N_5, N_3$

$\Rightarrow G = N_3 \dot{\times} N_5 \cong C_3 \times C_5$