The Sylow theorems

Def If G is a finite group, p a prime number

$p^\alpha \mid |G|$, but $p^{\alpha+1} \nmid |G|$, $H < G$ w/ $|H| = p^\alpha$

then we say H is a p-Sylow subgroup of G.

$Syl_p(G) = \{p\text{-sylow subgroups of } G\}$

$n_p = |Syl_p(G)|$.

$$inn_g : G \to G$$
$$h \mapsto ghg^{-1}$$

Theorem(s) (Sylow)

1. $Syl_p(G) \neq \emptyset$ ✓
2. $\forall P, Q \in Syl_p(G)$ then $\exists g \in G$ s.t. $gPg^{-1} = Q$ ✓
3. $n_p \equiv 1 \mod p$ ✓
4. $n_p \mid |G|$

Ex: if $|G| = 15$ 1. $\exists P_5, P_3 < G$ s.t. $|P_5| = 5$, $|P_3| = 3$

3. $n_3 \equiv 1 \mod 3 \Rightarrow n_3 = 1, 4, 7, 10, \ldots$

4. $n_3 \mid 15$

$\Rightarrow n_3 = 1$

2. $P_3 \triangleleft G$ gP_3g^{-1} is also a 3-Syl. subg.

since $n_3 = 1 \to$ "P_3

3. $n_5 \equiv 1 \mod 5$ $\quad n_5 = 1, 6, 11$

4. $n_5 \mid 15 \Rightarrow n_5 = 1$

$2 \Rightarrow P_5 \trianglelefteq G$

yesterday $\quad P_3, P_5 \trianglelefteq G \quad P_3 \cap P_5 = e$

$P_3 \cap P_5 < P_3, P_5$

order of P_3P_5 divisible by 3, 5

$|P_3 \cap P_5| \mid |P_3|, |P_5|$

$P_3, P_5 < P_3P_5$ is a subgroup $\quad \overset{\shortparallel}{1} \quad 3 \quad 5$

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$HN = \{hn \mid h \in H, n \in N\}$

not obviously closed

$(hn)(h'n') \rightsquigarrow h''n''$

$P_3P_5 = G$ $\quad$ yesterday $\Rightarrow G = P_3 \dot{\times} P_5$

$G \cong C_3 \times C_5$

First Sylow theorem $\quad (\mathrm{Syl}_p(G) \neq \emptyset)$

Use class equation:

$$|G| = |Z(G)| + \sum_{\substack{\text{some} \\ a \in G \\ \text{(one from each} \\ \text{non triv. conj class)}}} [G : C_G(a)]$$

size of $[a]$ conj class.

induct on $|G|$.

Case 1: $p \nmid |G| \Rightarrow Syl_p(G) = \{(e)\} \neq \emptyset$

Case 2: $p \mid |Z(G)|$ choose $g \in Z(G)$ $o(g) = p$

(by Cauchy's thm)

$\langle g \rangle \triangleleft G$ since

$$\underbrace{x\langle g\rangle x^{-1}}_{=\langle g \rangle} = \{x g^i x^{-1} \mid i\} = \{g^i \cancel{xx^{-1}} \mid i\} = \langle g \rangle$$

Consider $G/\langle g \rangle$

has smaller order, so

$|G| = p^\alpha m$

$\exists\, \overline{P} \in Syl_p(G/\langle g \rangle)$

but now: $P = \bigcup_{h\langle g\rangle \in \overline{P}} h\langle g \rangle$

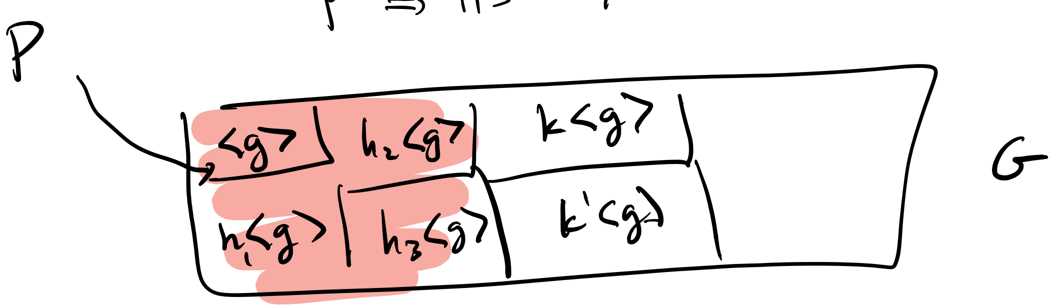
$|G/\langle g \rangle| = |G|/|\langle g \rangle| = p^{\alpha-1} m$

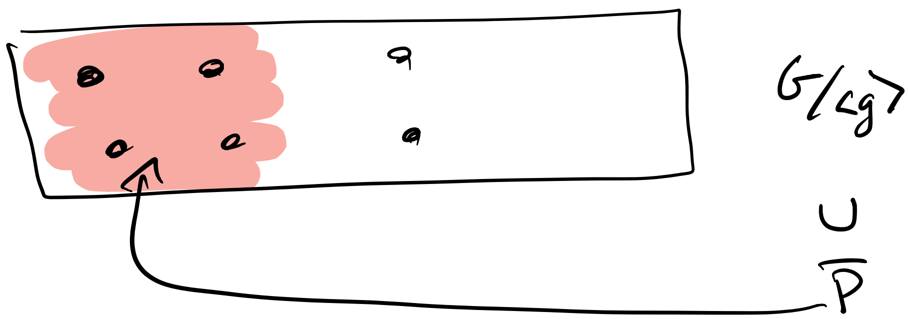
$|P| = |\overline{P}| \cdot (\text{size of cosets}) = p|\overline{P}|$

size of cosets $= |\langle g \rangle| = p$

$|\overline{P}| = p^{\alpha-1}$

$|P| = p^\alpha \Rightarrow$ it's a Sylow subgp.





Case 3: $p \mid |G|$, $p \nmid |Z(G)|$

Class eqn: $|G| = |Z(G)| + \sum_a [G:C_G(a)]$

if $p \mid [G:C_G(a)]$ all then

$p \mid |G| - \Sigma[\ \ ] = |Z(G)|$

but it doesn't

$\Rightarrow p \nmid [G:C_G(a)]$ some a.

But $C_G(a) \lneq G$ nontrivial subgp.

$\frac{|G|}{|C_G(a)|} = [G:C_G(a)]$ $\quad p \nmid [G:C_G(a)]$

$\Rightarrow$ same power of p divides $|G|$ & $|C_G(a)|$

$|G| = [G:C_G(a)]\,|C_G(a)|$

$p^\alpha \mid |C_G(a)|$

but $C_G(a) \lneq G \Rightarrow$ by induction $\exists$

$P < C_G(a)$ $\quad |P| = p^\alpha$

but $P < C_G(a) < G \Rightarrow P < G$. $\square$.

Let $P \in \mathrm{Syl}_p(G) \subset \mathcal{P}(G)$ $\quad P_i < G$

Consider it orbit $P_1 = P, P_2, \ldots, P_r$ under G via conj

i.e. $\{P_1, \ldots, P_r\} = \{gPg^{-1} \mid g \in G\}$

Suppose $Q < G$ $\quad |Q| = p^\beta$ some $\beta \le \alpha$ = max one.

then Q acts on $\{P_1, \ldots, P_r\}$ by conjugation.

$q \in Q \quad P_i \quad q \cdot P_i = qP_iq^{-1}$

suppose $P = P_1, P_2, \ldots, P_s$ is orbit of P under Q.

$s \le r$

Orbit Stab $\Rightarrow$ $s = \dfrac{|Q|}{|\mathrm{Stab}_Q P|}$ $\quad \mathrm{Stab}_Q P = Q \cap N_G(P)$

but $Q \cap N_G(P) < Q$ is a p-gp

$(Q \cap N_G(P))P$ is a subgp $\Rightarrow$ is also a p-gp.

(Consider in $N_G(P)$

$(Q \cap N_G(P)), P < N_G(P)$

but contains P which is a p-subgp of max'l size.

$$\Rightarrow (Q \cap N_G(P))P = P \Rightarrow$$

$$Q \cap N_G(P) \subset P, G$$

$$Q \cap N_G(P) \subset Q \cap P$$

$$N_G(P) \supset P$$

$$\Rightarrow \underline{Q \cap N_G(P) = Q \cap P}$$

$P = P_1, P_2, \dots, P_s$ orbit of P via Q action (conjugation)

$$s = \frac{|Q|}{|\mathrm{Stab}_G P|} \qquad \mathrm{Stab}_G P = Q \cap N_G(P) = Q \cap P \;!$$

$$s = [Q : Q \cap P] \qquad \text{Set } Q = P$$

$$s = [P : P] = 1$$

but any other orbit of $\underset{P}{\overset{}{Q}}$ ($Q = P$) acting on $\mathrm{Syl}_p(G)$

$$P_1', P_2', \dots, P_s'$$

$$s = \frac{|Q|}{|\mathrm{Stab}_G \cdot|} = [P : P_1' \cap P]$$

($|Q|$: $Q = P$)

not equal by laws

$$\Rightarrow p \mid s$$

~~We just showed:~~ $\{P_1,\dots,P_r\}$

$Syl_p(G)$ is union of orbits under P under conj

one orbit $\{P\}$ has size 1, other orbits have size a mult of p.

$$r \equiv_p 1.$$

$\Rightarrow$

<u>Next:</u> ~~Sylow $2\frac{1}{2}$: if $Q<G$ $|Q|=p^\beta$ then $Q \subset P_i$ some p-Sylow P_i.~~ if $P \in Syl_p(G)$ and $Q<G$ w/ $|Q|=p^\beta$ then $gQg^{-1} \subset P$ some g.

<u>Pf:</u> by contradiction, suppose $Q \not\subset P_i$ $\{P_1,\dots,P_r\} \subseteq Syl_p(G)$

then orbit of P_i under Q as above has size $s = \frac{|Q|}{|Q \cap P_i|}$ are all mults of p.

but all orbits size mult of p $\Rightarrow$ whole set $\{P_1,\dots,P_r\}$ has size a mult of p.

but $r \equiv_p 1$ ↯

therefore $Q \subset P_i$ some i.

$P_i = gPg^{-1}$

Z = every p-Sylow is conj.

Aside:

given $H < G$

consider $\underset{conj}{\mathrm{Stab}_G}(H) = \{g \in G \mid gHg^{-1} = H\}$

if this was the whole grp, then $H \triangleleft G$

always have H in

in general, define $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$

$$H \triangleleft N_G(H) < G$$

Why are all p-Sylows conj?