Recall: Sylow Theorems

G a finite group $\quad |G| = p^\alpha m \quad p \nmid m$

$Syl_p(G) = \{P < G \mid |P| = p^\alpha\}$ "the p-Sylow subgroups"

$n_p(G) = |Syl_p(G)|$

Theorem(s)

1. $Syl_p(G) \neq \emptyset$

2. $P, Q \in Syl_p(G)$ then $\exists g \in G$ s.t. $gPg^{-1} = Q$

3. $n_p \equiv 1 \mod p$

4. $\boxed{n_p \mid \cancel{|G|}\ m}$

Pf overview:

1: via induction & using class eqn & Cayley's thm

either: $\alpha = 0$ ✓

$p \mid |Z(G)| \Rightarrow$ get normal subgp of order p $\langle g \rangle$ induct via $G/\langle g \rangle$

$p \nmid |Z(G)|$ showed $p \nmid [G : C_G(a)]$

$\Rightarrow p^\alpha \mid |C_G(a)| \quad C_G(a) \lneq G$

(strong) induct via e.g. $C_G(a)$

Reminder:

Considered conjugates of some $P \in Syl_p(G)$

$$\{gPg^{-1} \mid g \in G\} = P_1, \dots, P_r$$

$$\boxed{(\stackrel{!}{=} Syl_p(G)) \qquad r = n_p!}$$

want

Consider Q some p-Sylow subgp.

we want to show: Q is on the list $P_1, \dots, P_r$

we also want $r = n_p \equiv 1 \bmod p$.

Observation: $r \mid |G|$ because $P_1, \dots, P_r$ are single orbit for G under conj.-action.

$\{P_1, \dots, P_r\}$ orbit of P_1 under G

$2 \Rightarrow 4 \qquad\qquad 2 \Leftarrow (r = n_p)$

$r = \dfrac{|G|}{|Stab_G P_1|}$ and $P_1 < Stab_G P_1 \qquad |Stab| = p^\alpha l \qquad r = \dfrac{p^\alpha m}{p^\alpha l}$

if Q acts on $P_1, \dots, P_r$ via conjugation $\qquad l = \dfrac{m}{r}$

$\boxed{r \mid m}$

critical observation: if Q fixes some P_i then $Q = P_i$.

Q fixes P_i

$\Updownarrow$

$Q \subset N_G(P_i)$

thus in G for P_i

$Q \cap N_G(P_i) \triangleleft Q$

i.e. an orbit can only have size 1 if it is $\{Q\}$

all other orbits have size mult. of p (orbit-stab)

$Q \ldots$

a p-gp

$Q = P_1 \rightsquigarrow Q$ fixes P_1, none of the others

$(Q \cap N_G(P_i))P_i = P_i$

$\Downarrow$

$Q \cap N_G(P_i) = Q \cap P_i$

$\cap P_i$

$P_1, \ldots, P_r$ union of orbits, all mult. of p except 1.

$\Rightarrow r \equiv 1 \mod p$ $(2 \Rightarrow 3)$

Q any p-Sylow if $Q \neq P_i$ any of's then it stabilizes none of them - all orbits size > 1 (mult. of p)

$\Rightarrow r$ mult of p ↯

$\Rightarrow$ stabilizes one $\Rightarrow$ it is one of them

$\Rightarrow Syl_p(G) = \{P_1, \ldots, P_r\}$

Q arbitrary in ↗ $n_p = r$ $\square$

Q acts on set $\{P_1, \ldots, P_r\}$ (r elements)

by $q \cdot P_i = qP_iq^{-1}$

orbit of P_i under action of Q =

$\{qP_iq^{-1} \mid q \in Q\}$

$\cap$

$Syl_p(G)$

is an orbit of Q so size divides $|Q|$

$\{P_1, \ldots, P_r\}$

is a disjoint union of these

$$|\{qP_iq^{-1} \mid q \in Q\}| = \begin{cases} 1 & \text{if } Q = P_i \\ \text{mult of } p & \text{else.} \end{cases}$$

a number, not 1, dividing $|Q| = p^{\alpha}$.

Correspondence theorem

G a gp, $N \triangleleft G$, $\overline{G} = G/N$

then given $H < G$, with $N \subset H$ then $N \triangleleft H$

and we can consider H/N.

if we are given $K < \overline{G}$ some subgp

can consider $\tilde{K} \equiv \{h \in G \mid hN \in K\} < G$

Theorem is: get a bijective correspondence

$$\left\{\begin{matrix}\text{subgroups of} \\ G \text{ containing } N\end{matrix}\right\} \qquad \left\{\begin{matrix}\text{subgroups} \\ \text{of } G/N\end{matrix}\right\}$$

$$N < H < G \longmapsto H/N < G/N$$

$$\tilde{K} < G \longleftarrow K < G/N$$

i.e. $\widetilde{\left(H/N\right)} = H$ if $N < H < G$

and $K = \tilde{K}/N$ if $K < G/N$

$$\left(\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd} \right)$$

<u>Furthr:</u>

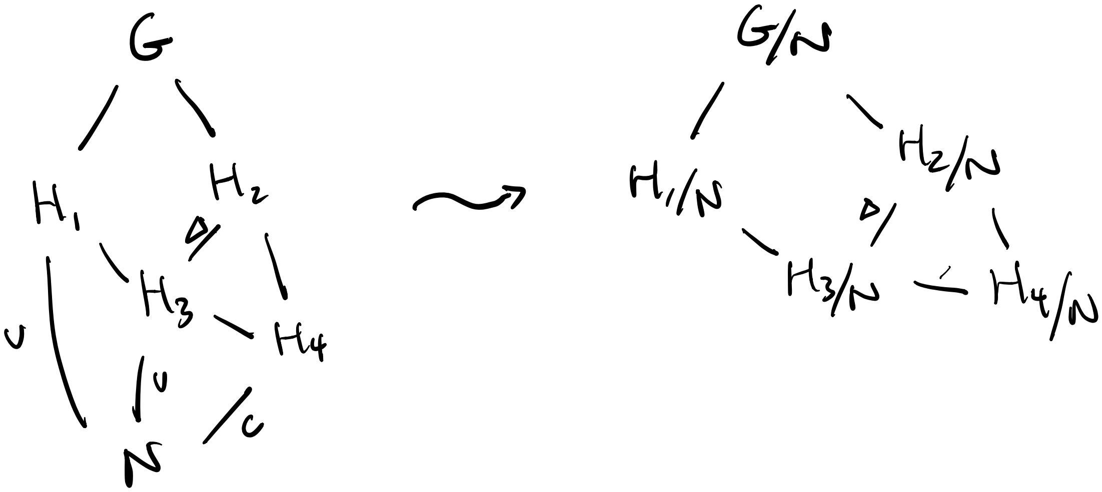
given $N < H_1, H_2 < G$

then $H_1 \subset H_2 \iff H_1/N \subset H_2/N$

and $H_1 \lhd H_2 \iff H_1/N \lhd H_2/N$

and

$$H_2/H_1 \cong \left(H_2/N\right) \big/ \left(H_1/N\right)$$



Final Sylow thought

if $P \in Syl_p(G)$ an $n_p(G) = 1$

then $P \trianglelefteq G$.

$gPg^{-1} \in Syl_p(G) \implies gPg^{-1} = P$.

$|G| = 18$ $\quad$ $P_3 \in Syl_3(G)$ $\quad$ $|P_3| = 9$.

$P_3 \trianglelefteq G$ $\quad$ $P_3 \overset{?}{\cong} C_3 \times C_3$ or C_9

$$\left.\begin{array}{l} n_3 \mid \cancel{18}\ 2 \\ n_3 \equiv 1 \mod 3 \end{array}\right\} \implies n_3 = 1.$$

"Fundamental theorem of Abelian gps"

$\Downarrow$

every finitely gen. Ab. gp. is a product of cyclic gps.

$|G| = 21$ $\qquad P_7 \trianglelefteq G$ $\qquad G/P_7 \cong C_3$

$\left.\begin{array}{l} n_7 \equiv 1 \bmod 7 \\ n_7 \mid 3 \end{array}\right\} n_7 = 1$ $\qquad P_7 \cong C_7$

$|G| = 30$ $\qquad \left.\begin{array}{l} n_5 \equiv 1 \ (5) \\ n_5 \mid 6 \end{array}\right\} n_5 = 1, 6$

$\left.\begin{array}{l} n_3 \equiv 1 \ (3) \\ n_3 \mid 10 \end{array}\right\} n_3 = 1, 10$

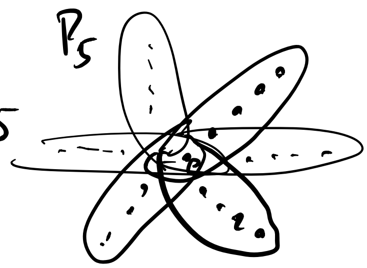
$\left.\begin{array}{l} n_2 \equiv 1 \ (2) \\ n_2 \mid 15 \end{array}\right\} n_2 = 1, 3, 5, 15$

Claim either $n_5 = 1$ or $n_3 = 1$

contradiction if $n_5 = 6$, $n_3 = 10$

1 elmt order 1 $\qquad P_5$ $\qquad P_5'$

$24 = 4 \cdot 6$ elements of order 5

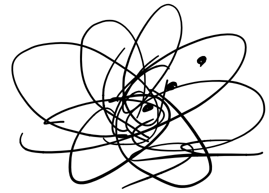


$P_5 \cap P_5' < P_5, P_5'$

order 5

$|P_5 \cap P_5'| \mid 5$

P_3 P_3'



20 elts ord 3

1 elt 4

Sylow tricks

- play numerology, try to get $n_p = 1$ some p.
- if doesn't work and n_p's large, try to count elements & run out of room.
- if some n_p is very small...