G = symmetries (D10)

$D_5 \triangleleft G$ $\qquad$ $C_{10} \triangleleft G$

$C_{10} = \langle \gamma \rangle$ $\qquad$ $\gamma \notin D_5$

Conj classes

sizes $\quad 1 \quad 2 \quad 2 \quad 5$

$\gamma D_5 \neq D_5$

$G = D_5 \cup \gamma D_5$

<u>Reminder</u>

Sylow theorems $\qquad$ G finite gp $|G| = p^\alpha m$, $p \nmid m$.

$$Syl_p(G) = \{ P < G \mid |P| = p^\alpha \}$$

$n_p = |Syl_p(G)|$

1) $n_p \neq 0$

2) $P, P' \in Syl_p(G) \Rightarrow P' = gPg^{-1}$ some $g \in G$ (sylow subgps are a single orbit under conjugation)

3) $n_p \equiv 1 \pmod p$

4) $n_p \mid m$

If $P \in Syl_p(G)$, $n_p = 1$ then $P \triangleleft G$.

Recall: if G is a p-group $|G| = p^\alpha$

then $Z(G) \neq \{e\}$

so $Z(G) \triangleleft G$ nontriv. normal subgp

or $Z(G) = G$

(G Abelian)

exercise: if $|G| = 8$ $|Z(G)| = 4$ then

G is Abelian

in fact if $G/Z(G)$ is cyclic

$\Rightarrow G$ Abelian.

$g \notin Z(G)$ s.t. $gZ(G)$ generates

$G = \{g^i z \mid z \in Z(G)\} \ldots$

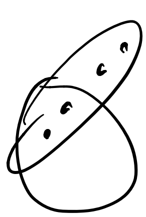
$|G| = 12$

$p = 2, 3$

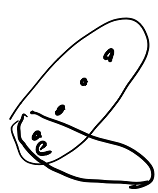
$n_2 \equiv 1 \mod 2$ $\quad n_2 \mid 3$

$n_3 \equiv 1 \mod 3$ $\quad n_3 \mid 4$

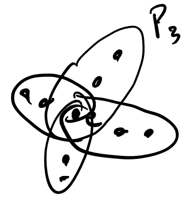
$12 = 2^2 \cdot 3$



P_2



P_3



what if have 4 P_3's

elmts of order 3 $= 8$

elmts of order 1 $= 1$

Strategy

1. numerology of n_p congruences
2. count & run out of elements (contradiction)
   good if n_p's are either 1 or pretty big.
3. Act on $Syl_p(G)$ works if some n_p is small.

$n_2 = 1$ or 3 3 small . . .

assume $n_2 = 3$ G acts on $Syl_2(G)$ by conjugation $|Syl_2(G)| = 3$

$g \cdot P = gPg^{-1}$ action is nontrivial because single orbit!

$$G \xrightarrow{\varphi} S_{Syl_2(G)} = S_3 \qquad \begin{bmatrix} P_1 & P_2 & P_3 \\ \downarrow & \downarrow & \downarrow \\ gP_1g^{-1} & gP_2g^{-1} & \vdots \end{bmatrix}$$

$$g \longmapsto [P \mapsto gPg^{-1}]$$

get a nontrivial hom $\varphi : G \to S_3$

$\ker\varphi \neq G$ $G/\ker\varphi \cong \operatorname{im}\varphi < S_3$

$|S_3| = 6$

$$1,2,3,6 = |\operatorname{im}\varphi| \,\big|\, 6 \qquad \frac{|G| = 12}{|\ker\varphi|} = 1,2,3,6$$

$$\Rightarrow |\ker\varphi| = \cancel{12}, 6, 4, 2$$

$\Rightarrow \ker\varphi \neq (e)$ since $|\ker\varphi| = 2, 4$ or 6

$|G| = 12$ ✓

Some famous results

Burnside's $q^a p^b$ theorem

If G is divisible by at most 2 primes then G is solvable ($\Rightarrow$ has a nontriv normal subgp)

solvable

$N \lhd G$

G/N Abelian

& N solvable

(e) is solvable.

Theorem Feit–Thompson

Every finite simple nonAbelian gp has even order.

"semidirect products"