Today: Semidirect Products

(putting groups together from subgroups)

Goal: Given $H, K < G$ when is $HK = G$ and moreover elements of $G$ uniquely represented as products $g = hk$ in this case, how can we describe the group structure?

Natural conditions: if $K \triangleleft G$, $H < G$ then $HK$ is always a subgroup.

$$HK = \{hk \mid h \in H, k \in K\}$$

Assumption 1: $K \triangleleft G$

Assumption 2: $HK = G$

Assumption 3: $H \cap K = (e)$

$K \triangleleft G$

$$hkh'k' = hh'h'^{-1}kh'k' = \underbrace{hh'}_{\in H}\overbrace{\underbrace{(h'^{-1}kh')}_{\in K}k'}^{\in K} \checkmark$$

$\quad\quad\quad\quad\quad H K \quad H K$

uniqueness: $\quad h \cdot e = e \cdot k$ shouldn't happen unless $h = k$

$\Rightarrow$

$A3 \Rightarrow \quad hk = h'k'$

$h'^{-1} h k = k'$

$\underbrace{h'^{-1} h}_{\in H} = \underbrace{k' k^{-1}}_{\in K} \Rightarrow h'^{-1} h = e \Rightarrow h = h'$

$\quad\quad\quad\quad\quad\quad\quad\quad k' k^{-1} = e \Rightarrow k' = k$

__Def__ If $H < G$, $K \triangleleft G$ with $HK = G$ & $H \cap K = (e)$ we say $G$ is an internal semidirect product of $H$ & $K$ write $G = K \dot{\rtimes} H$

$K \triangleleft$

$> H$

__External semidirect products__

Given groups $H, K$ and a homomorphism $\varphi : H \to \mathrm{Aut}(K)$. We define

$K \rtimes_\varphi H = K \times H$ as a set.

product:

$$(k,h)\cdot(k',h') = \big(k\,\varphi(h)(k'),\, hh'\big)$$

$$k\underbrace{h k'}_{}h' \qquad k(k' \text{ altered by } h)\, hh'$$

$$khk'h' = k\,(hk'h^{-1})\,h\,h'$$

$k'$ altered by $h$

(in internal case, $\varphi : H \longrightarrow \operatorname{Aut} K$

$h \longmapsto [k \mapsto hkh^{-1}]$

<u>Theorem</u> If $G = K \dot{\rtimes} H$, $K \trianglelefteq G$, $H < G$

then $G \cong K \rtimes_\varphi H$ where $\varphi : H \to \operatorname{Aut}(K)$

defined by $\varphi(h)(k) = hkh^{-1}$.

<u>Ex:</u> $|G| = 14$

$n_7 \equiv 1 \mod 7 \qquad n_2 \mid 2 \implies n_2 = 1$

$P_7 \trianglelefteq G$

$P_2 \qquad n_2 \equiv 1 \bmod 2 \qquad n_2 \mid 7 \qquad 1, 7$

$\underset{H}{\underset{\shortparallel}{P_2}} < G \qquad \underset{K}{\underset{\shortparallel}{P_7}} \triangleleft G \qquad |P_2 \cap P_7| \,\Big|\, \underset{2}{|P_2|}, \underset{7}{|P_7|}$

$\Rightarrow P_2 \cap P_7 = (e)$

$\underset{G}{\underset{\shortparallel}{P_2 P_7}} < G \qquad \underset{2,\ 7}{|P_7|, |P_2|} \,\Big|\, |P_2 P_7| \,\Big|\, \underset{14}{|G|}$

Poof $\Longrightarrow G = P_7 \rtimes P_2 \simeq P_7 \rtimes_\varphi P_2$

for some $\varphi : \underset{\langle \tau \rangle}{\underset{\shortparallel}{P_2}} \to \mathrm{Aut}(P_7)$

$\tau \overset{\varphi}{\longmapsto} \varphi(\tau) = f$

$\tau^2 = e$

$P_7 = \langle \sigma \rangle \circlearrowleft f$

$\sigma^7 = e$

$\underset{e}{\underset{\shortparallel}{\tau^2}} \longmapsto \underset{\underset{e}{\shortparallel}}{\underset{\varphi(\tau)\varphi(\tau)}{\underset{\shortparallel}{\varphi(\tau^2)}}}$

$f(\sigma) = \sigma^i$

$ff(\sigma) = \sigma^{i^2}$

$\sigma \underset{id}{\longmapsto} \sigma^{i^2} = \sigma^1$

$\sigma \longmapsto \sigma^i \longmapsto (\sigma^i)^i$

$i^2 \equiv 1 \bmod 7$

| | |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |
| 3 | 9 = 2 |
| 4 | 16 = 2 |
| 5 | 25 = 4 |
| 6 | 36 = 1 |

$\sigma \longmapsto \sigma^6 = \sigma^{-1}$

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Conclusion:

$\varphi(\tau)(\sigma) = \sigma$ or $\varphi(\tau)(\sigma) = \sigma^{-1}$

$\tau^i \sigma^j$

$KH$

$H = \{e, \tau\}$

$K = \{e, \sigma, \dots, \sigma^6\}$

$(\sigma^i \tau^j)(\sigma^{i'} \tau^{j'})$

$\tau \; \sigma$

$\tau\sigma = \varphi(\tau)(\sigma) \cdot \tau$

$= \sigma\tau$

Abelian, $C_2 \times C_7$

$\cong$

$C_{14}$

$\tau\sigma = \underbrace{\varphi(\tau)(\sigma)}_{\sigma^{-1}\tau}\tau$

$D_7$

$$C_{14} \longrightarrow C_2 \times C_7$$

$|G| = 18$ $\qquad P_3 \quad 9 \qquad C_3 \times C_3 \quad C_9$

$\qquad\qquad\qquad P_2 \quad 2$

$$P_2 \longrightarrow \mathrm{Aut}\, P_3$$

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$N \trianglelefteq G$

$$\begin{tikzcd} N & G/N \arrow[dl, bend right, dashed, "s"'] \\ G \arrow[ur, "\pi"] & \end{tikzcd} \qquad g \mapsto gN$$

$$G \cong N \rtimes_\varphi \overline{G} \iff \exists\, s : G/N \to G \text{ hom.}$$

s.t. $\pi \circ s = \mathrm{id}_{G/N}$

<u>ex:</u> $\quad C_2 \triangleleft C_4$

$$C_4 \simeq C_2 \rtimes_\varphi C_2$$