Forms on vector spaces

$$\begin{tikzcd} & \text{Bilinear forms} \arrow[dl] \arrow[dr] & \\ \text{Symmetric} & & \text{Alternating / Skew} \end{tikzcd}$$

(dot product)

Hermitian forms
(aka sesquilinear forms)

Quadratic forms

Def If V is a vector space over a field F
a bilinear form $b : V \times V \longrightarrow F$
such that it is linear in both variables
i.e. $b(v, -)$ and $b(-, v)$ are linear maps from V to F for any $v \in V$.

$$b(v, w_1 + w_2) = b(v, w_1) + b(v, w_2)$$

$$b(v, \lambda w) = \lambda\, b(v, w) = b(\lambda v, w)$$

$$b(v_1 + v_2, w) = b(v_1, w) + b(v_2, w)$$

$$b(0, v) = b(0, v) + b(0, v)$$

$$0 = b(0, v)$$

"Recall" If V a vectr space over F,

Define $V^* = \{T : V \to F \mid T \text{ a lnr transformation}\}$

if $f, g \in V^*$ "dual vectors"

$$\left.\begin{aligned}(f+g)v &\equiv f(v)+g(v)\\ (\lambda f)(v) &\equiv \lambda \cdot f(v)\end{aligned}\right\} V^* \text{ vectr space}/F.$$

For example $V = \{\text{polynomial functions } \mathbb{R} \to \mathbb{R}\}$

if $r \in \mathbb{R}$, get a lnr map $V \to \mathbb{R}$
$f \mapsto f(r)$

there's a natural map

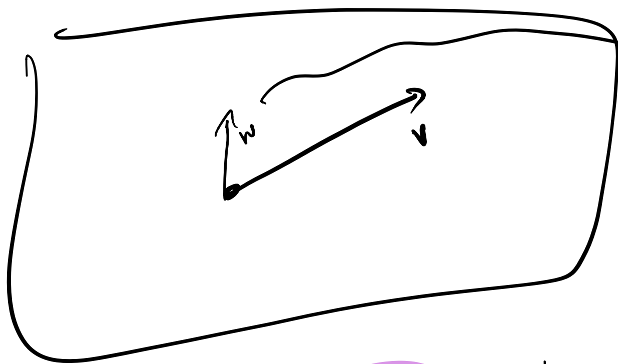
$$\mathbb{R} \to V^*$$
$$r \mapsto \underbrace{[f \mapsto f(r)]}_{\text{eval}_r}$$

$r_1, r_2 \longrightarrow$ same thg?

$\text{eval}_r \in V^* \qquad \text{eval}_r(f) \equiv f(r)$

If $b: V \times V \longrightarrow F$ bilinear form,

for $v \in V$, $b(v, -) \in V^* = \{\text{lin trans } V \to F\}$



$\# = w \cdot v$

therefore get a linear transformation

why?

$$V \longrightarrow V^*$$
$$v \longmapsto b(v, -)$$

if $V = F^n$ finite dim'l vector space basis $e_1, \ldots, e_n$

then $V^* = F^n$ w/ natural "dual" basis $f_1, \ldots, f_n$

<u>Def</u> $f_i(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$ $\qquad f_j = j^{th}$ coordinate function

$$f_j\left(\sum a_i e_i\right) = \sum a_i \underbrace{f_j(e_i)}_{0 \text{ except } j=i} = a_j \underbrace{f_j(e_j)}_{1} = a_j$$

$$b \longleftrightarrow (b_{ij})_{ij} = [B]_{\{e_i\}\{f_j\}} : V \longrightarrow V^*$$
$$v \longmapsto b(-,v)$$

where Bv coords are coeffs of $b(-,v)$

$$Be_k = (b_{ik})_i \longleftrightarrow \sum b_{ik} f_i$$

$$b(e_\ell, e_k) \;-\; \sum b_{ik} f_i(e_\ell) = b_{\ell k}$$
$$\parallel$$
$$e_\ell^t B e_k = b_{\ell k} \quad \text{ℓk matrix entry}$$

<u>ex:</u> std dot product:

$$B = (b_{ij}) \qquad b_{ij} = \langle e_i, e_j \rangle = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}$$

$$B = I_n$$

$b_1, \dots, b_4$

$\langle b_i, b_i \rangle = 1$ if $i = 1,2,3$

-1 if $i = 4$

$\langle b_i, b_j \rangle = 0$ if $i \neq j$

$$\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -1 \end{bmatrix}$$

$$b(e_1, e_2) = 1 \qquad b(e_i, e_i) = 0$$
$$\parallel$$
$$b(e_2, e_1) \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

<u>Def</u> A bilinear form is symmetric if $b(v,w) = b(w,v)$ all v, w

is skew-symmetric if $b(v,w) = -b(w,v)$

is alternating if $b(v,v) = 0$ all v.

Symmetric $\longleftrightarrow$ "inner products"

<u>ex:</u> skew, alternating: $b\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right) = \det\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right)$

<u>Claim:</u> If b is alternating, then it is skew

want to show: $b(v,w) = -b(w,v)$

$$\cancel{b(v+w, v+w)} = b(v, v+w) + b(w, v+w)$$

$$= \cancel{b(v,v)} + b(v,w) + b(w,v) + \cancel{b(w,w)}$$

alternating $\Rightarrow$ $b(v+w, v+w) = 0 = b(v,v)$
$= b(w,w)$

$$\Longrightarrow b(v,w) + b(w,v) = 0$$

$$b(v,w) = -b(w,v)$$

and if b is skew then it is alternating since:

$$b(v,v) = -b(v,v) \Rightarrow 2b(v,v) = 0$$

$$\nearrow b(v,v) = 0$$

$\|$

if

$\Downarrow$ [char $\neq 2$]

$$b(v,v) + b(v,w) = 0$$

$$g^2 = g \cdot g = e \qquad g^2 = e$$
$$g = e.$$

if F has char $= 2$

$$v + v = 2v$$

$$v + v = 1 \cdot v + 1 \cdot v = \underbrace{(1+1)}_{\in F} \cdot v = \underset{\in F}{(2)} \cdot v = 0$$

$$\| $$

$$1 \cdot (v+v)$$

<u>Char 2:</u> symm = alternating $\Rightarrow$ skew

<u>char $\neq 2$:</u> sym $\neq$ alternating $\Leftrightarrow$ skew

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<u>Rem:</u> if char $F \neq 2$ and b = bilin form / V f.dim'l

then b can be uniquely written as $b_0 + b_1$ $\quad b_0 = \text{sym}$, $b_1 = \text{skew}$.

<u>Proof</u> Note $\{\text{bilin forms}\} \simeq M_n(F)$

$\text{Bil}(V) \quad V = F^n$

$e_i(B+B')e_j$

$= e_iBe_j + e_iB'e_j$

let $\sigma : \text{Bil}(V) \longrightarrow \text{Bil}(V)$

$b \longmapsto \sigma b$

$\sigma b(v,w) = b(w,v)$

$\sigma^2 = id \rightsquigarrow$ Jordan form σ is diagonal

w/ eigvals ± 1

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}^2 = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

$\lambda_i = \pm 1$

eval $1 \Rightarrow \sigma b = b$

eval $-1 \Rightarrow \sigma b = -b$

$$\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & \end{bmatrix}$$

$$\begin{bmatrix} 0 & & -1 \\ & \ddots & \\ 1 & & 0 \end{bmatrix}$$