Last time F - field V/F vector space.

Defined bilinear form: to be a map

$$b: V \times V \longrightarrow F \quad \text{linear in both coordinates}$$

i.e. $b(v,-): V \longrightarrow F$
$w \mapsto b(v,w)$

$b(-,v)$
both linear, all $v \in V$.

b is • symmetric if
$$b(v,w) = b(w,v)$$

• skew if
$$b(v,w) = -b(w,v)$$

• alternating if
$$b(v,v) = 0 \quad \text{all } v$$

char $\neq 2$
sym $\neq$ skew $\Longleftrightarrow$ alt.

char $= 2$
sym $=$ skew $\Longleftarrow$ alt.

if b is a bilinear form then it induces a linear map

$$\varphi_b: V \longrightarrow V^* = \{\text{Lin trans } V \to F\}$$
$$v \longmapsto b(v,-)$$

we say b is <u>nondegenerate</u> on the left. if this map $V \to V^*$ is an isomorphism

(V is finite dim'l)

Recall: "standard" inner product $V = F^n$

basis $e_1, \dots, e_n$

$$b\left(\sum x_i e_i, \sum y_i e_i\right) = \sum x_i y_i$$

$$\vec{x} = \sum x_i e_i = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$b(x,y) = x^t y$

consequence

if $T : F^n \to F^n$ any lin trans.

$$b(Tx, y) = (Tx)^t y = x^t T^t y$$
$$= x^t (T^t y)$$
$$= b(x, T^t y)$$

Consequence $\Rightarrow$ if T preserves b. (i.e. $b(x,y) = b(Tx, Ty)$

Then $b(x,y) = b(Tx, Ty)$

for all x, y $= b(x, T^t T y)$

for y, let x vary, $b(-, y) = b(-, T^t T y)$ all y

all x, y $b(x,y) = b(x, T^t T y)$

$\Leftrightarrow$ (nondeg) $y = T^t T y$ all y

$\downarrow$

$\Leftrightarrow T^t T = I_n.$

all x,y $b(x,y) = b(Tx, Ty)$

Standard def $O_n = \{ T \in GL_n(\mathbb{R}) \mid T^t T = I_n \}$

Adjoints

If b is a nondegenerate bilinear form, either sym or skew

then for a linear transformation $T: V \to V$,

we define the adjoint of T with respect to b

to be the linear trans T^b s.t.

$$b(Tx, y) = b(x, T^b y)$$

Prop T^b exists and is unique

ex: if b is the standard inner prod on F^n, then $T^b = T^t$.

How to define T^b ?

given $y \in V$ want to define $T^b y$

$T^b y$ is determined by the linear map $b(-, T^b y)$

since b is skew or sym & nondegen.

but we want $\quad b(x, T^\flat y) = b(Tx, y)$

so $\quad b(-, T^\flat y) = b(T-, y)$

we define $T^\flat y$ to be the vector s.t. $b(x, T^\flat y) = b(Tx, y)$

$$T^\flat y = \varphi_b^{-1}(b(T-, y))$$

$$T^\flat(y_1+y_2) \overset{?}{=} T^\flat y_1 + T^\flat y_2$$

these are equal, enough to show (by nondegeneracy)

$$b(-, T^\flat(y_1+y_2)) = b(-, T^\flat y_1 + T^\flat y_2)$$

$$T^\flat(y_1+y_2) = T^\flat y_1 + T^\flat y_2 \iff \varphi(T^\flat(y_1+y_2)) = \varphi(T^\flat y_1 + T^\flat y_2)$$

i.e. $\quad b(x, T^\flat(y_1+y_2)) = b(x, T^\flat y_1 + T^\flat y_2)$ all x

<u>Properties</u>: if b as above (skew or symmetric)

$(T_1 + T_2)^\flat = T_1^\flat + T_2^\flat$ (homomorphism)

$(T_1 T_2)^\flat = T_2^\flat T_1^\flat$ (anti-homomorphism)

$(T^\flat)^\flat = T$ (involution)

$e_1 \quad e_2$

$b(e_1, -) = 0$

$b(e_2, e_1) = 1$

$b(e_2, e_2) = 0.$

$b(e_1, -)$

$b(e_1 + e_2, -)$

weird example to keep in mind.

might recall from last time:

$b \longleftrightarrow M_b$

& choice of basis

$b(x,y) = x^t M_b y$

$(M_b)_{ij} = b(e_i, e_j)$

$$b(Tx,y) = (Tx)^t M_b y = b(x, \underbrace{T^b}_{?} y)$$

$$= x^t T^t M_b y$$

$$= x^t (M_b M_b^{-1}) T^t M_b y$$

$$= x^t M_b \underbrace{(M_b^{-1} T^t M_b)} y$$

formula for adj $\boxed{T^b = M_b^{-1} T^t M_b}$

<u>Conclusion</u>: given a bilinear form b

can consider the linear transformations T s.t.

$$b(Tv, Tw) = b(v, w)$$

as before, these are the matrices s.t.

$$T^b T = I_n.$$

<u>Def</u> $O(b) \equiv \{ T \in GL(V) \mid T^b T = I_n \}$

this is a group, $= \{ T \in GL(V) \mid b(v,w) = b(Tv, Tw) \}$

<u>examples</u> $F = \mathbb{R}$ $b =$ std. inner product. on $\mathbb{R}^n$

$\rightsquigarrow O(b) = O_n$

if $F = \mathbb{R}$ $b =$ "sympectic inner prod"
namely skew form

$$M = \begin{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & & & \\ & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & & \\ & & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \\ & & & \ddots \end{bmatrix}$$

$O(b) = SP_{2n}$
$\mathbb{R}^{2n}$

basis $e_1, f_1, e_2, f_2, \ldots, e_n, f_n$
$b(e_i, e_j) = 0 = b(f_i, f_j)$

$b(e_i, f_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & i = j \end{cases}$

$b(f_j, e_i) = -b(e_i, f_j)$

$GL(V)$ general linear gp

$SL(V)$ = determinant 1

$\parallel$

SL_n $n = \dim V$.

Hermitian inner products

Def A sesquilinear form on a complex vector space V is a map $V \times V \xrightarrow{h} \mathbb{C}$ s.t. $h(v, -)$ linear and $h(-, w)$ is conjugate-linear

$T : V \to V$ $\mathbb{C}$ spaces is conj linear if

$T(v+w) = T(v) + T(w)$

$T(\lambda v) = \bar{\lambda} T(v)$

If $h(v,w) = \overline{h(w,v)}$ "Hermitian"

$h(v,w) = -\overline{h(w,v)}$ "skew Hermitian"

$SU(2)$ SO_3