So far: given a vector space V over a field F

can define notions of

skew/symm/alt. bilinear forms

$\rightsquigarrow$ can consider the subgp of $GL(V)$

which preserves the form

i.e. $b(Tv, Tw) = b(v, w)$

Important forms

$F = \mathbb{R}$ standard inner product (w/r/to a basis)

O_n $O(n)$

F general, symm bilin form b O_b $O(b)$

if form is standard inner prod for general field F

$O_n(F)$

F general, standard skew form on $2n$ dim'l v.space

basis $e_1, f_1, e_2, f_2, \ldots, e_n, f_n$

$\omega(e_i, f_i) = 1 = -\omega(f_i, e_i)$

others are 0. PSp

$Sp_{2n}(F)$ Sp_{2n} ~~SSp~~

Hermitian forms $\mathbb{C} > \mathbb{R}$

$V / \mathbb{C}$ complex v.space

$h : V \times V \longrightarrow \mathbb{C}$ <u>sesquilinear</u>

$$h(\lambda v, w) = \bar{\lambda} h(v, w)$$
$$h(v, \lambda w) = \lambda h(v, w)$$

as before, $T \in GL(V)$ then $\exists ! \, T^*$ $\quad T^\dagger$

s.t. $h(Tv, w) = h(v, T^* w)$

can consider lin transformations s.t.

$$h(v, w) = h(Tv, Tw)$$

group: $U(h)$

standard herm inner product.

$$h\left(\sum a_i e_i , \sum b_i e_i\right) = \sum \bar{a}_i b_i$$

U_n

Prefixes

S, P

special = determinant 1

projective = mod out by scalar matrices.

$$SL_n = \{T \in GL_n \mid \det T = 1\}$$

GL_n

$\cup$

diagonal scalar matrices

$$GL_n / \text{scalar} = PGL_n$$

$$\begin{bmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{bmatrix} \subset Z(GL_n) \lhd GL_n$$

$$PSL_n = SL_n / \text{scalars of det } 1$$

some scalar matrices $\subset SL_n$

$$\begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{bmatrix} \quad \text{s.t. } \det = 1$$
$$\lambda^n = 1$$

$$\begin{tikzcd} SL_n \arrow[r, hook] \arrow[d] \arrow[dr] & GL_n \arrow[d] \\ PSL_n \arrow[r, hook] & PGL_n \end{tikzcd}$$

$$\underset{\pm 1}{\text{scalars}} \subset O_n(F) \qquad\qquad O_6$$
$$\cup \qquad\qquad\qquad \cup$$
$$SO_n(F) \qquad\qquad SO_6$$

subgp s.t. det = 1

$$O_n(F) \xrightarrow{\det} F^\times$$

$$T \in O_n \Leftrightarrow T^t T = Id$$

$$\begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{bmatrix}^2 = Id \qquad \lambda = \pm 1$$

$$PO_n = O_n / \underset{\text{are } \pm 1}{\text{scalars}} \qquad PO_6 = O_6/\text{scalars.}$$

$$PSO_n = \begin{cases} SO_n/\pm 1 & n \text{ even} \\ SO_n & n \text{ odd} \end{cases}$$

$SO_n \supset$ scalars

$$\begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{bmatrix}^2 = I$$

and det 1 $\Rightarrow \lambda^n = 1$

$\lambda = 1$ or -1

SU_n $\qquad PSU_n$

PU_n

Side comment:

Can form unitary groups with quadratic field extensions in char $\neq 2$

F a field, $L \supset F$ is another field, then we say L is quadratic if $\dim_F L = 2$ and that elmts in L can be uniquely written as $a + b\sqrt{d}$, some $d \in F$ not a square in F

$$(a+b\sqrt{d})(a'+b'\sqrt{d}) =$$
$$(aa' + bb'd) + (ab' + b'a)\sqrt{d}$$

$\boxed{1, \sqrt{d} \text{ basis.}}$

conjugation $\overline{(a+b\sqrt{d})} = a - b\sqrt{d}$

given L/F quad ext, $L = \mathbb{Q}(\sqrt{2})$ $F = \mathbb{Q}$

V/L vector space, can talk about hermitian forms w/r/to L/F, $\bar{\ }$

$$U_n \quad SU_n \quad PU_n \quad PSU_n$$

$F = \mathbb{F}_3 = \{\bar{0}, \bar{1}, \bar{2}\} = \{\bar{0}, \bar{1}, -\bar{1}\}$

$\mathbb{F}_9 = L = \mathbb{F}_3(i) = \{a+bi \mid a, b \in \mathbb{F}_3\}$

finite field w/ 9 elements.

$\mathbb{Z}/9\mathbb{Z} \not\cong \mathbb{F}_9$.

<u>skip ahead</u>: for any number $q = p^n$ p prime $n \geq 1$

$\exists!$ field of order q, denoted $\mathbb{F}_q$

Classification of finite gps

PS( ) tends to be simple.

<u>example</u> $PSL_n(\mathbb{F}_q)$ always simple if $n \geq 3$

also simple if $n = 2$, $q \geq 4$

$PSp_{2n}(\mathbb{F}_q)$ simple $n \geq 3$;

or $n = 2$, $q \geq 3$;

$n = 1$, $q \geq 4$

$PSO_n(\mathbb{F}_q)$ simple if stuff is big enough.

variation if $q = 2^n$

det $\longleftrightarrow$ "Arf invariant"

on the 5 (unit of currency)

Turkish bill

These matrix gps above are part of a classification of matrix gps ($\leftrightarrow$ classification of Lie gps)

two broad cats of matrix gps

Classical types

lin. trans which preserve bilinear / hermitian form

$A_\ell \longleftrightarrow PSL_{\ell+1}$

$B_\ell \longleftrightarrow PSO_{2\ell+1}$

$C_\ell \longleftrightarrow PSp_{2\ell}$

$D_\ell \longleftrightarrow PSO_{2\ell}$

Exceptional types

E_6 $\qquad E_7(\mathbb{F}_8)$

E_7

E_8

F_4

G_2

symmetries of an octonion algebra
8 dim'l algebra
(vector space w/ "mult" structure)
nonassociative

symmetries of the Albert algebra
27 dim'l non assoc. alg.
"Jordan algebra smallest exceptional Jordan alg"

$^2A_\ell$ $^2D_\ell$ $^3D_4$

$^2E_6$ $^2F_4$ ^2B $^2G_2$