What are we trying to classify?

Topological groups

G top space

$(G, \cdot)$ group

$$\left.\begin{array}{l} G \times G \xrightarrow{m} G \\ G \xrightarrow{i} G \\ g \longmapsto g^{-1} \end{array}\right\} \text{cont. maps.}$$

locally compact groups

$\mathbb{Q}_p$, $\mathbb{Q}_p^\times$

locally compact abelian groups

Lie groups (manifold-groups)

$\mathbb{C}^n$

Complex Lie groups ($\mathbb{C}$-manifolds)

Matrix Groups

"Linear Algebraic groups"

solns to poly eqns in $GL_n(F)$

F some field

$F = \mathbb{R}$

$F = \mathbb{C}$

Today:

SU_2, SO_3, $SL_2\mathbb{R}$, $SL_2\mathbb{C}$

- $SU_2 = \left\{ \underset{\overset{\shortparallel}{T}}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \in GL_2(\mathbb{C}) \;\middle|\; T^*T = I,\ \det T = 1 \right\}$

$= \{ T : \mathbb{C}^2 \to \mathbb{C}^2 \mid$ preserve $\overset{\text{standard}}{\vee}$ hermitian form $\}$

$\det T = 1 \iff$ cofactor matrix = inverse matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$/\!/$$

$$T^* = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \qquad T^*T = I$$

$$\bar{a} = d \quad \bar{c} = -b$$

$$\bar{b} = -c \quad \bar{d} = a$$

$$T = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \qquad \det T = 1$$

$$a\bar{a} + b\bar{b} = 1$$

$$SU_2 = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in GL_2(\mathbb{C}) \;\middle|\; a\bar{a} + b\bar{b} = 1 \right\}$$

bijection

$a = a_0 + a_1 i$

$b = b_0 + b_1 i$

$a\bar{a} = a_0^2 + a_1^2$

$b\bar{b} = b_0^2 + b_1^2$

$$\{ (a,b) \in \mathbb{C}^2 \mid a_0^2 + a_1^2 + b_0^2 + b_1^2 = 1 \}$$

$$\mathbb{R}^3 \cup \{pt\} = S^3 = \{ (a_0, a_1, b_0, b_1) \in \mathbb{R}^4 \mid \quad " \quad \}$$

$$\left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in M_2(\mathbb{C}) \;\middle|\; a, b \in \mathbb{C} \right\}$$

$$\mathbb{H} = \left\langle \underset{1}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{i}{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}, \underset{j}{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}, \underset{k}{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}} \right\rangle_{\mathbb{R}}$$

<u>real span</u>: matrices of the form

$q = a + bi + cj + dk$ these multiply like <u>quaternions</u>.

$ij = -ji = k$

$\mathbb{H}^\times = \mathbb{H} \setminus \{0\}$

as before, if $v = xi + yj + zk$

$v \longmapsto qvq^{-1}$ is a rotation, all rotations in SO_3 can be described in this way.

$\square^\times = \{$gp under mult of nonzero stuff$\}$

recall: $\bar{q} = a - bi - cj - dk$

and $q\bar{q} = a^2 + b^2 + c^2 + d^2 = |q|^2$

and $q^{-1} = \frac{1}{|q|^2} \cdot \bar{q}$

SU_2 is in bijection w/ $\{ q \in \mathbb{H} \mid q\bar{q} = 1 \}$

= isomorphism of groups!

$\cap$ $\mathbb{H}^\times$

give rotations.

preserve Hermitian forms on $\mathbb{C}^2$

$$q \xmapsto{R} (v \mapsto q v q^{-1})$$

rotation

$R(q)$

it turns out

$$R(q) = R(q') \text{ iff } q = \lambda q' \text{ some } \lambda \in \mathbb{R}^\times$$

mult gp of $\neq 0$

$$\mathbb{H}^\times \longrightarrow SO_3 \quad \text{surjective.}$$
$$q \longmapsto R(q)$$

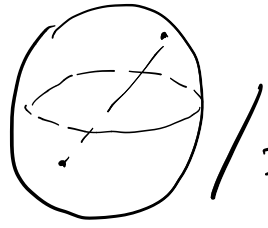
better, we restrict to length 1 quaternions

$$SU_2 = \{ q \in \mathbb{H}^\times \mid |q|^2 = 1 \} \xrightarrow{R} SO_3$$

$= S^3$

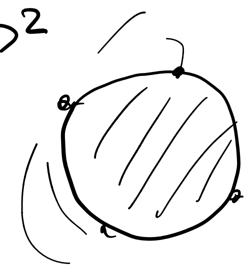
$$R(q) = R(q') \Leftrightarrow q = \pm q'$$

$$\ker R = \pm 1$$

 $/\pm 1$

identify antipodal pts

$\mathbb{RP}^2$



$$S^3/\pm 1 = \mathbb{RP}^3$$

$$\shortparallel$$

$$SU_2/\pm 1 \cong SO_3$$

$\uparrow$ 1st iso

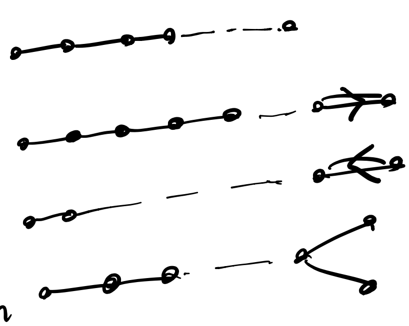
$SU_2 = Spin_3$

in general, $\forall n \geq 3$ have gps

$$Spin_n \longrightarrow SO_n$$

2-1 maps (kernel $= \pm 1$)

Classification of simple complex Lie gps via "types" corresp. to diagrams "Dynkin diagrams"



SL_{n+1} A_n

$Spin_{2n+1}$ B_n

Sp_{2n} C_n

$Spin_{2n}$ D_n

- B_1
- A_1

$Spin_3(\mathbb{C})$

$SL_2(\mathbb{C})$

Spin$_3$

SU$_2$

D_2 $\qquad$ $A_1 \times A_1$

$|(a,b,c,d)|^2$

$a^2+b^2+c^2-d^2$

Spin$_{3,1}$ $\qquad$ $SL_2\mathbb{C}$

$SO_{3,1}$

$SL_2\mathbb{R} \times SL_2\mathbb{R}$

Spin$_4(\mathbb{C})$ $\qquad$ $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$

- A_1 $\qquad$ $SL_2(\mathbb{C})$ $\qquad$ $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$ $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} a & 0 & 0 \\ & b & 0 \\ & & (ab)^{-1} \end{bmatrix} \quad \begin{bmatrix} 1 & a & \\ & 1 & b \\ & & 1 \end{bmatrix}$$