

Groups "in nature" come in two principal flavors

- finite groups (mostly what we will consider)
- infinite / continuous groups (other courses)

⋮  
⋮

Examples:

$GL_n(\mathbb{C})$   $n \times n$  invertible matrices under multiplication.  
(noncommutative)

various subgroups

$GL_n(\mathbb{R})$        $GL_n(\mathbb{Q})$

$ab = ba$   
not always true.

$$O_n(\mathbb{R}) = \{ T \in GL_n(\mathbb{R}) \mid TT^t = I \}$$

$$O_n(\mathbb{C}) = \begin{matrix} \cdot & \cdot & - & \mathbb{C} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \otimes & \cdot & \cdot & \cdot \end{matrix}$$

$$O_n(\mathbb{Q}) = \cdot \quad \cdot \quad \cdot \quad \otimes \quad \cdot \quad \cdot$$

Operator  $*$ ,  $+$  conjugate transpose

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$$

$$U_n = \{ T \in GL_n(\mathbb{C}) \mid TT^* = I \}$$

$$U_1 = \{ T \in GL_1(\mathbb{C}) \mid TT^* = I \}$$
$$\{ t \in \mathbb{C}^\times \mid t\bar{t} = 1 \} = S^1$$

Klein-four group  $V \subset GL_2(\mathbb{R})$

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$$

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma\tau = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

	$e$	$\sigma$	$\tau$	$\sigma\tau$
$e$	$e$	$\sigma$	$\tau$	$\sigma\tau$
$\sigma$	$\sigma$	$e$	$\sigma\tau$	$\tau$
$\tau$	$\tau$	$\sigma\tau$	$e$	$\sigma$
$\sigma\tau$	$\sigma\tau$	$\tau$	$\sigma$	$e$

commutative  
(Abelian)

Def A group is called Abelian if it satisfies the commutative law.

$S_3$  is nonabelian  $\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\sigma\tau \neq \tau\sigma$$

Example: Quaternion group:  $Q_8$

$$1, i, j, k, -1, -i, -j, -k \quad \text{w/ mult rule} \quad i^2 = j^2 = k^2 = -1 \quad ij = k$$

$$\text{ex: } j(-k) = -jk = (-j)j = ijj = -i$$

alternately: explicitly consider matrices in  $GL_2(\mathbb{Q})$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

in  $\mathbb{Q}_8$ , have  $\{\pm 1\}$  as a subgroup

$$\{1, i, i^2, i^3\} = \{1, i, -1, -i\} \subset \mathbb{Q}_8$$

### Isomorphisms

Def we say that two groups  $G, G'$  are isomorphic if  $\exists$  a bijective map  $\varphi: G \rightarrow G'$  such that mult. is preserved/respected  
i.e.  $\varphi(gh) = \varphi(g)\varphi(h)$

[  
ex: if  $V$  is vector space it is also a group w/r respect to addition. and isomorphisms of vector spaces  $\rightsquigarrow$  isom. of groups]

we call a bijective map  $\varphi$  as above an isomorphism from  $G$  to  $G'$ .

Def A homomorphism between groups  $G \xrightarrow{\varphi} G'$  is any map  $\varphi: G \rightarrow G'$  such that

$$\varphi(gh) = \varphi(g)\varphi(h)$$

Proposition: if  $\varphi: G \rightarrow G'$  is a homomorphism

then  $\varphi(e_G) = e_{G'}$

Pf:  $\varphi(e) = \varphi(ee) = \varphi(e)\varphi(e)$

$\left. \begin{matrix} \text{hom.} \\ \text{mult.} \\ \text{by} \\ (\varphi(e))^{-1} \\ \text{on left.} \end{matrix} \right\}$

$$\cancel{\varphi(e)^{-1}\varphi(e)} = \cancel{\varphi(e)^{-1}} \cdot \varphi(e)\varphi(e)$$

$$e = e\varphi(e)$$

$$\boxed{e = \varphi(e)}$$

exercise: show  $\varphi(g^{-1}) = \varphi(g)^{-1}$  if  $\varphi$  is a hom.

$$e = gg^{-1}$$

$$e = \varphi(e) = \varphi(gg^{-1}) = \varphi(g)\varphi(g^{-1})$$

$$\varphi(g)^{-1}e \leq \cancel{\varphi(g)^{-1}} \varphi(g) \varphi(g^{-1})$$

$$\varphi(g)^{-1} = e \in \varphi(g^{-1}) = \varphi(g^{-1})$$


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$C_n$  = cyclic group of order  $n = \mathbb{Z}/n\mathbb{Z}$

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$$\{e, g, g^2, \dots, g^{n-1}\}$$

$$g^i g^j = \begin{cases} g^{i+j} & \text{if } i+j < n \\ g^{i+j-n} & \text{if } i+j \geq n \end{cases}$$

$$g^0 = e$$

if  $G$  is any group,  $g \in G$ , defined

$$\langle g \rangle = \{g^i \mid i \in \mathbb{Z}\}$$

we showed is a subgroup

order of  $g$  is smallest  $n \geq 0$  such that  $g^n = e$

$$\Rightarrow \langle g \rangle = \{e, g^1, g^2, \dots, g^{n-1}\}$$

and these are all distinct!

$$\forall i, j \in \mathbb{Z}, i \neq j \Rightarrow g^i \neq g^j$$

$$\begin{aligned}
 g^i &= gj \quad g^j(g^i)^{-1} = e \\
 &\Rightarrow g^j g^{-i} = e \\
 &\Rightarrow g^{j-i} = e \\
 0 \leq j-i &< n \\
 &\text{contradict} \\
 &\text{order } n.
 \end{aligned}$$

Cor: if  $g \in G$  has order  $n$ , then  
 $\langle g \rangle$  is isomorphic to  $C_n$

$$\begin{array}{c}
 \varphi: C_n \rightarrow \langle g \rangle \\
 o^i \mapsto g^i
 \end{array}
 \quad \left. \begin{array}{l} \text{this gives an isom} \\ \text{whenever } g \text{ has} \\ \text{order } n. \end{array} \right\}$$

Notation:  $G \approx G'$  means  $G$  isomorphic to  $G'$   
 (Artin  $\approx$ ) (other people use  $\cong$ )

Ram: if  $g \in G$  has infinite order ( $\nexists n \in \mathbb{N} \text{ s.t. } g^n = e$ )

$$\text{then } \langle g \rangle \cong \mathbb{Z}^+$$

$\mathbb{Z} \xrightarrow{g} \langle g \rangle$        $i \mapsto g^i$       bijective. (obviously surjective)

why injective?      if       $g^i = g^j$   
 $\Rightarrow g^{i-j} = e$

$$\Rightarrow i-j=0 \Rightarrow i=j$$

→ injective.

Def if  $G$  is a group, an automorphism of  $G$   
 is an isomorphism from  $G$  to itself.

Ex:  $C_3 = \langle e, \sigma, \sigma^2 \rangle \rightleftharpoons \begin{matrix} e \mapsto e \\ \sigma \mapsto \sigma^2 \\ \sigma^2 \mapsto \sigma \end{matrix}$

$$g = \overset{\circ}{\sigma^2} \quad \langle g \rangle = \langle e, g, g^2, g^3, \dots \rangle \\ = \langle e, \sigma^2, (\sigma^2)^2 \rangle = C_3$$

$$C_3 \xrightarrow{\sim} \langle g \rangle \xleftarrow{\sim} C_3$$

$e \mapsto e$   
 $\sigma \mapsto g$   
 $\sigma^2 \mapsto g^2$

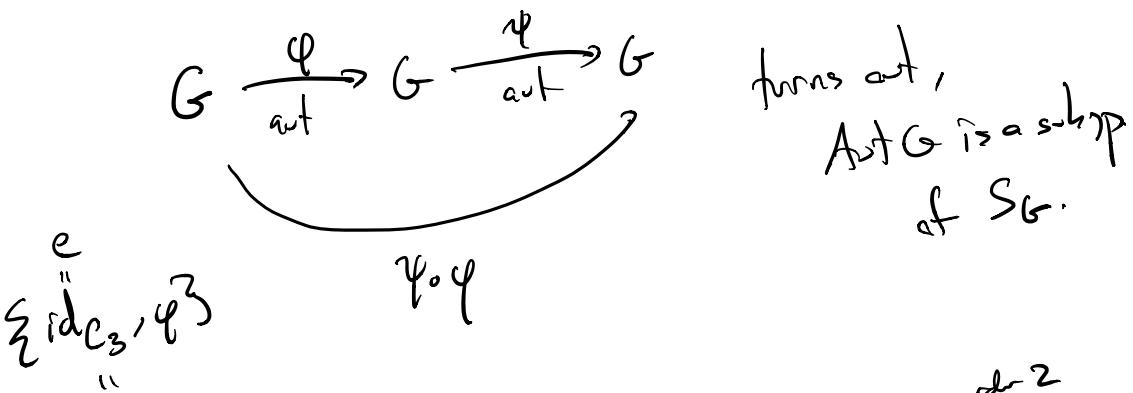
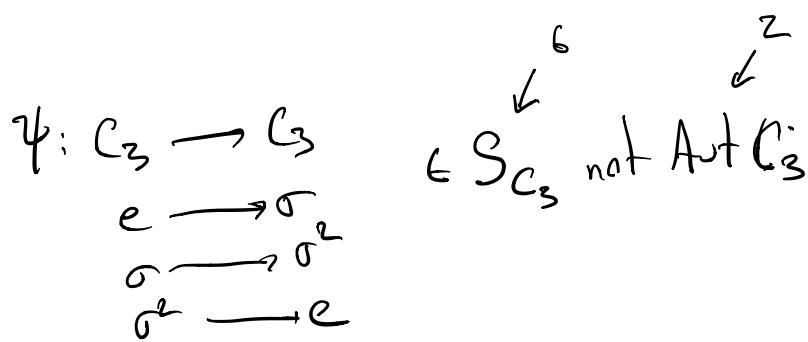
This is an isom

this  $\varphi$  is a natural aut. of  $C_3$

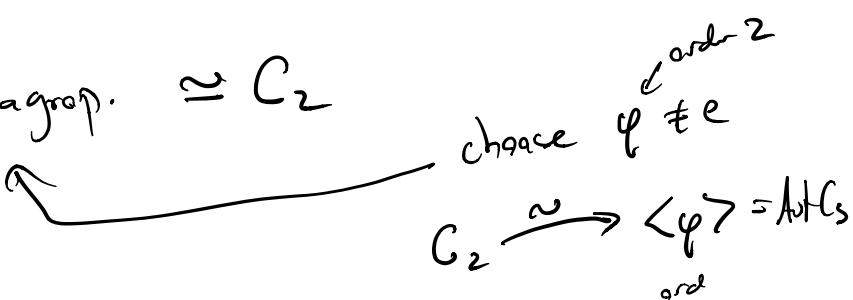
and it's the only one. (two nonidentity elmts are either permuted or swapped)

Note: If  $G$  is a group,  $\text{Aut } G$  is a group.

in fact,  $\text{Aut } G \subset S_G = \{ \varphi: G \rightarrow G \mid$   
bijective}



$\text{Aut } C_3$  is a group.  $\simeq C_2$



$$G \xrightarrow{\text{inn}} \text{Aut } G \quad \text{"conjugation"}$$

$g \mapsto \varphi_g = \text{inn}(g)$   
 $\varphi_g(h) = g^h g^{-1}$

Claim 1: this defines  
 an aut.  $\varphi_g$   
 Claim 2: this defines  
 a hom  $G \rightarrow \text{Aut } G$   
 $\text{inn}(gh) = \text{inn}(g)\text{inn}(h)$

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we are halfway into chap. 2.4