Groups "in nature" come in two principal flavors

- finite groups (mostly what we will consider)
- infinite / continuous groups (other courses)

⋮

Examples:

$GL_n(\mathbb{C})$ $n \times n$ invertible matrices under multiplication.

(noncommutative)

$ab = ba$ not always true.

various subgroups

$GL_n(\mathbb{R})$ $GL_n(\mathbb{Q})$

$O_n(\mathbb{R}) = \{T \in GL_n(\mathbb{R}) \mid TT^t = I\}$

$O_n(\mathbb{C}) = \quad " \quad \mathbb{C} \quad "$

$O_n(\mathbb{Q}) = \quad " \quad \mathbb{Q} \quad "$

Operation $*, \dagger$ conjugate transpose

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$$

$U_n = \{T \in GL_n(\mathbb{C}) \mid TT^* = I\}$

$$U_1 = \{T \in GL_1(\mathbb{C}) \mid TT^* = I\}$$
$$\{t \in \overset{\shortparallel}{\mathbb{C}^\times} \mid t\bar{t} = 1\} = S^1$$

Klein-four group $V \subset GL_2(\mathbb{R})$

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$$

$$e = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \quad \sigma = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \quad \tau = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$\sigma\tau = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$$

	e	σ	τ	$\sigma\tau$
e	e	σ	τ	$\sigma\tau$
σ	σ	e	$\sigma\tau$	τ
τ	τ	$\sigma\tau$	e	σ
$\sigma\tau$	$\sigma\tau$	τ	σ	e

commutative (Abelian)

<u>Def</u> A group is called Abelian if it satisfies the commutative law.

S_3 is nonabelian $\quad \tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\sigma\tau \neq \tau\sigma$$

<u>Example</u>: Quaternion group: Q_8

$1, i, j, k, -1, -i, -j, -k$ w/ mult rule $i^2 = j^2 = k^2 = -1$ $\quad ij = k = -ji$

ex: $j(-k) = -jk = (-j)ij = ijj = -i$

<u>alternately:</u> explicitly consider matrices in $GL_2(\mathbb{C})$

$$1 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \quad i = \begin{pmatrix} i & \\ & -i \end{pmatrix} \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

in Q_8, have $\{\pm 1\}$ as a subgroup

$$\{1, i, i^2, i^3\} = \{1, i, -1, -i\} < Q_8$$

Isomorphisms

<u>Def</u> we say that two groups G & G' are isomorphic if $\exists$ a bijective map $\varphi: G \longrightarrow G'$ such that mult. is preserved / respected

i.e. $\varphi(gh) = \varphi(g)\varphi(h)$

[<u>ex:</u> if V is vector space it is also a group w/ respect to addition. and isomorphisms of vector spaces $\rightsquigarrow$ isom.'s of groups]

we call a bijective map φ as above an <u>isomorphism</u> from G to G'.

Def A homomorphism between groups G & G' is any map $\varphi: G \longrightarrow G'$ such that

$$\varphi(gh) = \varphi(g)\varphi(h)$$

Proposition: if $\varphi: G \longrightarrow G'$ is a homomorphism then $\varphi(e_G) = e_{G'}$

hom.

Pf: $\varphi(e) = \varphi(ee) = \varphi(e)\varphi(e)$

mult. by $\varphi(e)^{-1}$ on left.

$$\cancel{\varphi(e)^{-1}\varphi(e)} = \cancel{\varphi(e)^{-1}\cdot}\varphi(e)\varphi(e)$$

$$e = e\varphi(e)$$

$$\boxed{e = \varphi(e)}$$

exercise: show $\varphi(g^{-1}) = \varphi(g)^{-1}$ if φ is a hom.

$$e = gg^{-1}$$

$$e = \varphi(e) = \varphi(gg^{-1}) = \varphi(g)\varphi(g^{-1})$$

$$\varphi(g)^{-1}e = \varphi(g)^{-1}\varphi(g)\varphi(g^{-1})$$

$$\varphi(g)^{-1} = e\,\varphi(g^{-1}) = \varphi(g^{-1})$$

C_n = cyclic group of order n = $\mathbb{Z}/n\mathbb{Z}$

$\quad \shortparallel$

$\{e, \sigma, \sigma^2, \ldots, \sigma^{n-1}\}$

$$\sigma^i \sigma^j = \begin{cases} \sigma^{i+j} & \text{if } i+j < n \\ \sigma^{i+j-n} & \text{if } i+j \geq n \end{cases}$$

$$\sigma^0 = e$$

if G is any group, $g \in G$, defined

$$\langle g \rangle = \{ g^i \mid i \in \mathbb{Z} \}$$

we showed is a subgroup

order of g is smallest $n > 0$ such that $g^n = e$

$$\Rightarrow \langle g \rangle = \{e, g^1, g^2, \ldots, g^{n-1}\}$$

and these are all distinct!

$i, j < n \quad g^i \neq g^j$

WLOG $i < j \Rightarrow$

$g^i = g^j \qquad g^j (g^i)^{-1} = e$

$\Rightarrow g^j g^{-i} = e$

$\Rightarrow g^{j-i} = e$

$0 \le j - i < n$

contradicting order $= n$.

<u>Cor</u>: if $g \in G$ has order n, then $\langle g \rangle$ is isomorphic to C_n

$$\left.\begin{aligned} \varphi : C_n &\longrightarrow \langle g \rangle \\ \sigma^i &\longmapsto g^i \end{aligned}\right\}$$ this gives an isom whenever g has order n.

<u>Notation</u>: $G \simeq G'$ means G isomorphic to G'

(Artin $\approx$) (other people use $\cong$)

<u>Rem</u>: if $g \in G$ has infinite order ($\nexists\, n \neq 0$ s.t. $g^n = e$)

then $\langle g \rangle \simeq \mathbb{Z}^+$

$$\mathbb{Z} \xrightarrow{\varphi} \langle g \rangle$$
$$i \longmapsto g^i$$

bijective. (obviously surjective)

why injective? if $g^i = g^j$

$\Rightarrow g^{i-j} = e$

$\Rightarrow i-j = 0 \Rightarrow i = j$

$\Rightarrow$ injective.

Def if G is a group, an automorphism of G is an isomorphism from G to itself.

Ex: $C_3 = \langle e, \sigma, \sigma^2 \rangle \circlearrowleft \varphi$

$$e \longmapsto e$$
$$\sigma \longmapsto \sigma^2$$
$$\sigma^2 \longmapsto \sigma$$

$\cup$

$g = \sigma^2$

$\langle g \rangle = \langle e, g, g^2, g^3, \dots \rangle$

$= \langle e, \sigma^2, \underset{\sigma}{\underset{\shortparallel}{(\sigma^2)^2}} \rangle = C_3$

$C_3 \xrightarrow{\sim} \langle g \rangle \overset{\shortparallel}{} C_3$

this is an isom

$$e \longmapsto e$$
$$\sigma \longmapsto g$$
$$\sigma^2 \longmapsto g^2$$

this φ is a nontrivial aut. of C_3

and it's the only one. (two nonidentity elmts are either preserved or swapped)

Note: If G is a group, $\operatorname{Aut} G$ is a group.

in fact, $\operatorname{Aut} G < S_G = \{\varphi : G \to G \mid \text{bijective}\}$

$$\begin{aligned} \psi : C_3 &\longrightarrow C_3 \\ e &\longrightarrow \sigma \\ \sigma &\longrightarrow \sigma^2 \\ \sigma^2 &\longrightarrow e \end{aligned} \quad \in \overset{6}{S_{C_3}} \text{ not } \overset{2}{\operatorname{Aut} C_3}$$

$$\begin{tikzcd} G \arrow[r, "\varphi", "\text{aut}"'] \arrow[rr, bend right, "\psi \circ \varphi"'] & G \arrow[r, "\psi", "\text{aut}"'] & G \end{tikzcd}$$

turns out, $\operatorname{Aut} G$ is a subgp of S_G.

$\{\overset{e}{\overset{\shortparallel}{\mathrm{id}_{C_3}}}, \varphi\}$

$\shortparallel$

$\operatorname{Aut} C_3$ is a group. $\simeq C_2$

choose $\varphi \neq e$ (order 2)

$C_2 \xrightarrow{\sim} \langle \varphi \rangle = \operatorname{Aut} C_3$

ord

$G \xrightarrow{\text{inn}} \mathrm{Aut}\, G$ "conjugation"

$g \longmapsto \varphi_g = \mathrm{inn}(g)$

$\varphi_g(h) \equiv g h g^{-1}$

Claim 1: this defines an aut. φ_g

Claim 2: this defines a hom $G \to \mathrm{Aut}\, G$

$\mathrm{inn}(gh) = \mathrm{inn}(g)\,\mathrm{inn}(h)$

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we are halfway into chap. 2.4