<u>Today</u> group actions (instead of equiv. relations)

- <u>Recall</u>. we discussed isomorphisms, homomorphisms

homomorphism $\varphi: G \longrightarrow H$

$$\varphi(gh) = \varphi(g)\varphi(h)$$

isomorphism = bijective homomorphism.

monomorphism = injective homomorphism

$K \subset G$, $S \subset H$, $\varphi: G \to H$, $\varphi|_K$, $\varphi|^S$

$\varphi: G \longrightarrow H$ monomorphism then

$\{$ G is in bijection with $\varphi(G) \subset H$

bijection.

$\varphi: G \to \underbrace{\varphi(G)}_{\text{im}\,\varphi} \subset H$ and in fact $\varphi(G) < H$

and $\varphi(G)$ is isomorphic to G.

automorphism = isomorphism from a group to itself.

$$\varphi: G \longrightarrow G$$

endomorphism = hom from a gp to itself

epimorphism = surj. hom.

Actions

Examples

$GL_n(\mathbb{C})$ acts on $\mathbb{C}^n$

$T \in GL_n(\mathbb{C})$, $v \in \mathbb{C}^n$ can define $T(v)$

$T \cdot v$

$T(Sv) = (TS)v$ $\quad Tv$

Ex $S_X = \{\text{bijective maps } \sigma: X \to X\}$

$(\sigma\tau)(x)$

S_X acts on X

$\sigma \in S_X$, $x \in X$

$\boxed{\sigma(x)}$

$\sigma \cdot x$

σx

$(\sigma\tau)\cdot x = \sigma\cdot(\tau\cdot x)$

$\sigma(\tau(x))$ by def of fun composition

Def A left ~~An~~ action of a group G on a set X is a rule

$$G \times X \longrightarrow X$$
$$(g, x) \longmapsto g(x)$$

$g \cdot x$

gx

such that

1) $e \cdot x = x$

2) $\sigma\cdot(\tau\cdot x) = (\sigma\tau)\cdot x$

(Alternately $G \times X \longrightarrow X$
$G \longrightarrow \mathrm{Fun}(X, X)$)

Examples

$\mathbb{Z}$ acting on $\mathbb{R}$

$$(n, x) \longmapsto n + x$$

$n(x) = n + x$

$(m+n)(x) = m + n + x$ ✓

$\overset{?}{=}$

$m(n(x)) = m + (n(x))$

$= m + n + x$

$0(x) = 0 + x = x.$

G any group,

G acts on G, by left multiplication.

$$g \cdot h \equiv gh \qquad e \cdot h = h$$

$$g\cdot(h\cdot k) = (gh)\cdot k$$

$$g\cdot(h \cdot k) = g \cdot hk = ghk \qquad (gh)\cdot k = ghk$$

G acts on G by conjugation:

$$g \cdot h = g h g^{-1}$$

$$g\cdot(h\cdot k) = gh \cdot k = gh(k)(gh)^{-1} = ghkh^{-1}g^{-1}$$

$$g\cdot(h\cdot k) = g(h\cdot k)g^{-1} = g(hkh^{-1})g^{-1} = ghkh^{-1}g^{-1}$$

<u>Notice</u>: if $g, h \in G$ $\quad (gh)^{-1} = h^{-1}g^{-1}$

(just check $(h^{-1}g^{-1})(gh) = e$

<u>ex</u>: (of ex)

$G = GL_n(\mathbb{R})$ $\qquad$ G acts on G

$$\overset{\in}{T} \cdot \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} Tv_1 & | & Tv_2 & \cdots \end{pmatrix}$$

$T \rightsquigarrow U T U^{-1}$ "change of basis action"

<u>Def</u> if G acts on X and $x \in X$, we define

$$Gx = \{ g \cdot x \mid g \in G \}$$ "orbit of x under the action"

$T \in GL_n(\mathbb{C})$ $\quad$ $GL_n(\mathbb{C})$ conj action, orbit of T

if $T = I_n$ $\quad$ $U I_n U^{-1} = I_n$

orbit is just $\{I_n\}$

In G acting on itself via left mult, single orbit = whole group. choose any $x \in G$

$$G \cdot x \subseteq G$$

$$g x^{-1} \cdot x \ni g$$

fundamental principle of group actions:

- If G acts on X then X is a disjoint union of orbits.
- orbits are either equal or non-intersecting.

i.e. orbits give an equivalence relation

def $x \sim y$ if $g \cdot x = y$ some g.

Pf: $\sim$ equiv reli

sym. $g \sim g$ ✓ because $e \cdot g = g$

reflex: if $x \sim y$ then by def $g \cdot x = y$

$$\Downarrow$$

$$g^{-1} \cdot (g \cdot x) = g^{-1} \cdot y$$

$$\shortparallel$$

$$(g^{-1} g) \cdot x$$

$$\shortparallel$$

$$e \cdot x = x$$

$$g^{-1} \cdot y = x$$

$$\Rightarrow y \sim x$$

transitivity:

if $x \sim y$ & $y \sim z$ then by def $g \cdot x = y$ some g

$h \cdot y = z$ some h

$$g \cdot x = y$$

$$\Downarrow$$

$$h \cdot (g \cdot x) = h \cdot y$$

$$(hg) \cdot x = h \cdot y = z$$

$$x \sim z$$

equiv. classes

$$[x] = \{ y \in X \mid x \sim y \} = \{ y \in X \mid gx = y \text{ some } g \in G \}$$

$$= \{ gx \mid g \in G \} = G \cdot x = Gx$$

orbit of x.

if $y = gx \Rightarrow [x] = [y]$

$Gx = Gy$

~~xf $f(x)$ $f\circ g(x)$ xfg~~

<u>Def</u> If G is a group, X is a set, a right action of G on X is a rule

$$X\times G \longrightarrow X$$

such that

$$(x,g) \longmapsto x\cdot g \qquad x\cdot e = x$$

$$xg \qquad x\cdot(gh) = (x\cdot g)\cdot h$$

$$x \longrightarrow xg$$

$GL_n(\mathbb{C})$ acts on $\mathbb{C}^n$ (on left)

$$T\cdot v = Tv$$

<u>Define</u>: $v\cdot T = Tv$

$$v\cdot(ST) = STv$$

$$\overset{?}{=}$$

$$(v\cdot S)\cdot T$$

$$= (Sv)\cdot T = TSv$$

$GL_n(\mathbb{C})$ acts on $\mathbb{C}^n$ on right via row vectors.

$$v^t\cdot T = v^t T$$

(right) orbits xG

Example: if $H < G$ then H acts on G on right & left

$$\begin{aligned} H \times G &\to G \\ (h,g) &\mapsto hg \end{aligned} \qquad \begin{aligned} G \times H &\to G \\ (g,h) &\mapsto gh \end{aligned}$$

orbits: $Hg = \{hg \mid h \in H\}$ $\qquad gH = \{gh \mid h \in H\}$

called right & left cosets of H in G.

Observation: for these actions all orbits have same size

$$H \to Hg, \quad h \mapsto hg \qquad H \to gH, \quad h \mapsto gh$$

$$xg^{-1} \leftarrow\!\!\!\rightarrow x$$

bijective

each orbit has form Hg and each Hg is in bijection w/ H.

In particular, if G is a finite group and $m = \#$ of left cosets of H in G

then each coset has $\#H$ elements.

$$\Rightarrow \#G = (\#H)m$$

Theorem (Lagrange's theorem)

if G is a finite group, $H < G \Rightarrow \#H \mid \#G$

Def if G acts on X

the kernel of the action $\equiv \{g \in G \mid g \cdot x = x \text{ all } x \in X\}$

ex: if $T = \lambda I_n \in GL_n(\mathbb{C})$ then $TST^{-1} = S$

$\Rightarrow T$ is in kernel of conj. action.

if $H < G$, H acts on left by mult.

then kernel is only identity:

$$h \cdot g = g \Rightarrow hg = g \Rightarrow hgg^{-1} = gg^{-1}$$

$$\Rightarrow h = e.$$