

Today group actions (instead of equiv. relations)

- Recall: we discussed isomorphisms, homomorphisms

homomorphism $\varphi: G \rightarrow H$
 $\varphi(gh) = \varphi(g)\varphi(h)$

isomorphism = bijective homomorphism.

monomorphism = injective homomorphism

$$\begin{array}{ccc} \begin{array}{c} K \\ \downarrow \\ \varphi: G \rightarrow H \\ \downarrow \\ \varphi|_K \end{array} & \begin{array}{c} S \\ \downarrow \\ \varphi|_S \end{array} & \\ & & \end{array}$$

$\varphi: G \rightarrow H$ monomorphism then $\{ G \text{ is in bijection with } \varphi(G) \subset H$

$\varphi: G \xrightarrow{\text{bijection}} \varphi(G) \subset H$ and in fact $\varphi(G) < H$
 and $\varphi(G)$ is isomorphic to G .

automorphism = isomorphism from a group to itself.

$$\varphi: G \rightarrow G$$

endomorphism = hom from a gp to itself

epimorphism = surj. hom.

Actions

Examples

$GL_n(\mathbb{R})$ acts on \mathbb{R}^n

can define $T(v)$

$$T(Sv) = (TS)v \quad \begin{array}{c} \nearrow \\ T \cdot v \\ \downarrow \\ Tv \end{array}$$

Ex $S_X = \{\text{bijective maps } \sigma: X \rightarrow X\}$

S_X acts on X
 $\sigma \quad x$

$\sigma(x)$
 $\sigma \cdot x$
 σx

$(\sigma\tau)(x)$
 $(\sigma\tau) \cdot x$
 $\sigma(\tau \cdot x)$
 by def of function composition

Def A left action of a group G on a set X is a rule

$G \times X \rightarrow X$

$(g, x) \mapsto g(x)$
 $g \cdot x$
 gx

such that

1) $e \cdot x = x$

2) $\sigma \cdot (\tau \cdot x) = (\sigma\tau) \cdot x$

(Alternatively $G \times X \rightarrow X$
 $G \rightarrow \text{Fun}(X, X)$)

Examples

\mathbb{Z} acting on \mathbb{R}

$(n, x) \mapsto n+x$

$n(x) = n+x$

$(m+n)(x) = m+n+x$

|| ?

$m(n(x)) = m+(n(x))$

$= m+n+x$

$0(x) = 0+x = x$

G any group,

G acts on G , by left multiplication.

$$g \cdot h \equiv gh$$

$$e \cdot h = h$$

$$g \cdot (h \cdot k) = (gh) \cdot k$$

$$g \cdot hk$$

$$ghk$$

$$ghk$$

G acts on G by conjugation:

$$g \cdot h = ghg^{-1}$$

$$g \cdot (h \cdot k) = gh \cdot k = gh(k)(gh)^{-1} = ghkh^{-1}g^{-1}$$

$$g \cdot (h \cdot k) \cdot g^{-1} = g(hkh^{-1})g^{-1}$$

Notice: if $g, h \in G$ $(gh)^{-1} = h^{-1}g^{-1}$

(just check $(h^{-1}g^{-1})(gh) = e$)

ex: (of ex)

$$G = GL_n(\mathbb{C})$$

G acts on G

$$T \cdot \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ Tv_1 & Tv_2 & \dots & \\ | & | & & | \end{pmatrix}$$

$$T \mapsto U T U^{-1} \text{ "change of basis action"}$$

Def if G acts on X and $x \in X$, we define

$$Gx = \{gx \mid g \in G\} \quad \text{"orbit of } x \text{ under the action"}$$

$T \in GL_n(\mathbb{C})$ $GL_n(\mathbb{C})$ conj action,
orbit of T

if $T = I_n$ $U I_n U^{-1} = I_n$
orbit is just $\{I_n\}$

In G acts on itself via left mult,
single orbit = whole group. choose $x = e \in G$

$$G \cdot x = G$$
$$gx^{-1} \cdot x = g$$

fundamental principle of group actions:

- If G acts on X then X is a disjoint union of orbits.
- orbits are either equal or non-intersecting.

i.e. orbits give an equivalence relation

def $x \sim y$ if $g \cdot x = y$ some g .

Pf: \sim equiv. rel.

sym. $g \sim g$ ✓ because $e \cdot g = g$

reflex: if $x \sim y$ then by def $g \cdot x = y$

$$\Downarrow \\ g^{-1} \cdot (g \cdot x) = g^{-1} \cdot y$$

$$\Downarrow \\ (g^{-1}g) \cdot x \\ e \cdot x = x$$

$$g^{-1} \cdot y = x \\ \Downarrow \\ x \sim x$$

transitivity:

if $x \sim y$ & $y \sim z$ then by def $g \cdot x = y$ some g
 $h \cdot y = z$ some h

$$g \cdot x = y$$

$$\Downarrow \\ h \cdot (g \cdot x) = h \cdot y$$

$$(hg) \cdot x = h \cdot y = z$$

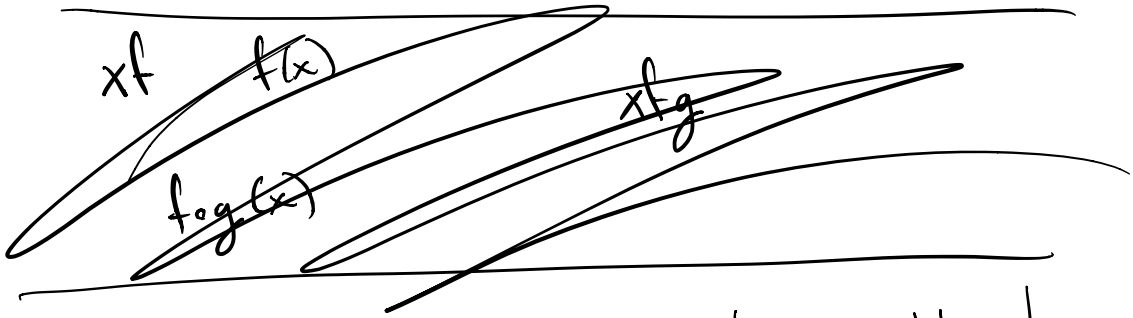
$$x \sim z$$

equiv. classes

$$[x] = \{ y \in X \mid x \sim y \} = \{ y \in X \mid g \cdot x = y \text{ some } g \in G \}$$

$$= \{ g \cdot x \mid g \in G \} = G \cdot x = Gx \\ \text{orbit of } x.$$

$$\text{if } y = g \cdot x \Rightarrow [x] = [y] \\ Gx = Gy$$



Def If G is a group, X is a set, a right action of G on X is a rule

$$\begin{array}{l}
 X \times G \longrightarrow X \\
 (x, g) \longmapsto x \cdot g \\
 x \longmapsto xg
 \end{array}
 \quad \text{such that}
 \quad
 \begin{array}{l}
 x \cdot e = x \\
 x \cdot (gh) = (x \cdot g) \cdot h
 \end{array}$$

$GL_n(\mathbb{C})$ acts on \mathbb{C}^n (on left)

$$T \cdot v = Tv$$

Defn: $v \cdot T = Tv$

$$v \cdot (ST) = STv$$

$$\begin{array}{l}
 \text{"?"} \\
 (v \cdot S) \cdot T \\
 \text{"} \\
 (Sv) \cdot T = TSv
 \end{array}
 \quad \times$$

$GL_n(\mathbb{C})$ acts on \mathbb{C}^n on right via row vectors.

$$v^t \cdot T = v^t T$$

(right) orbits xG

Example: if $H < G$ then H acts on G on right & left

$$H \times G \rightarrow G$$

$$(h, g) \mapsto hg$$

$$G \times H \rightarrow G$$

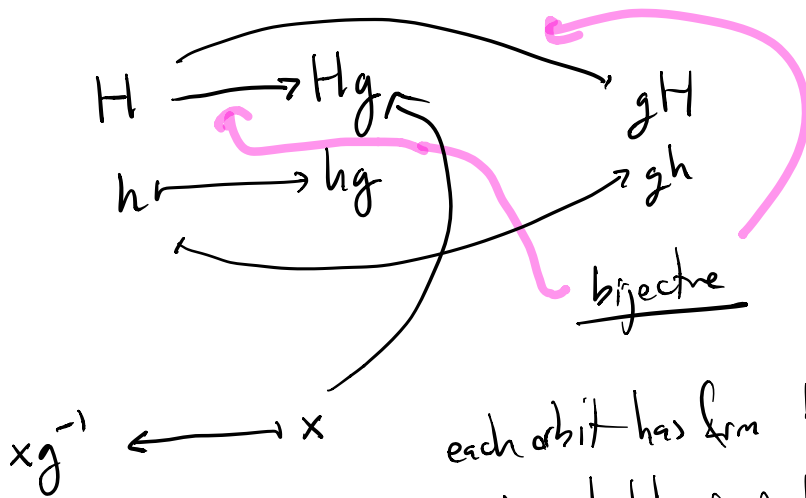
$$(g, h) \mapsto gh$$

orbits: Hg
 $\{hg \mid h \in H\}$
 called right orbit

gH
 $\{gh \mid h \in H\}$
 left orbit

cosets of H in G .

Observation: for these actions all orbits have same size



each orbit has form Hg
 and each Hg is in bijection
 w/ H .

In particular, if G is a finite group and $m = \#$ of cosets
 of H in G

then each coset has $\#H$ elements.

$$\Rightarrow \#G = (\#H)m$$

Theorem (Lagrange's theorem)

if G is a finite group, $H < G \Rightarrow \#H \mid \#G$

Def if G acts on X

the kernel of the action $\equiv \{g \in G \mid g \cdot x = x \text{ all } x \in X\}$

ex if $T = \lambda I_n \in GL_n(\mathbb{C})$ then $TST^{-1} = S$
 $\Rightarrow T$ is in kernel of conj. action.

if $H < G$, H acts on left by mult.

then kernel is only identity:

$$h \cdot g = g \Rightarrow hg = g \Rightarrow hgg^{-1} = gg^{-1} \\ \Rightarrow h = e.$$