Recap:

Defined the notion of a group action on a set

G acts on X means

$$G \times X \xrightarrow{\alpha} X \quad \text{s.t.} \quad e \cdot x = x$$

$$(g,x) \longmapsto \alpha(g,x) \qquad g \cdot (h \cdot x) = (gh) \cdot x$$

$$g \cdot x$$

S_X acts on X

$X = G$

G acts on itself in more than one interesting way

left mult. action

$$g \cdot x = gx$$

conj. action

$$g \cdot x = gxg^{-1}$$

$H < G$ H acts on G via left mult. $h \cdot x = hx$

Also defined right actions similarly

Default: Action = left action

H acts as left mult.

orbit of some $g \in G$ is Hg "a right coset of H in G"

the right coset of g

H acts on right,

orbits gH "left cosets"

Noticed: these all have same size, $|H| = \#H$

$$\Rightarrow \quad |G| = |H| \underbrace{\#\text{cosets of } H \text{ in } G}_{[G:H]}$$

$$[G:H] = \frac{|G|}{|H|}$$ integer "Lagrange's theorem"

Theorem "Orbit-Stabilizer"

if G acts on a set X and $x \in X$

then $$|Gx| = \frac{|G|}{|\mathrm{Stab}(x)|}$$

Def If G acts on X, $x \in X$ $\quad \mathrm{Stab}(x) = \{g \in G \mid gx = x\}$

ex: if G acts on G via conjugation then

$$\text{Stab}(e) = \{g \in G \mid geg^{-1} = e\}$$

$$= G$$

if G acts on G via left mult then

$$\text{Stab}(x) = \{g \in G \mid gx = x\} = \{e\}$$

$$g = e$$

if $G = S_n$ acting on $\{1, \dots, n\}$

$$\text{Stab}(n) = \{\sigma : \{1, \dots, n\} \circlearrowleft \mid \sigma(n) = n\}$$

$$\text{"}\|\text{"}$$

$$S_{n-1}$$

Note: $\text{Stab}(x) < G$ (always a subgp)

Note: $\ker(\alpha) = \bigcap_{x \in X} \text{Stab}(x)$

α = the action

Theorem "Orbit-Stabilizer"

if G acts on a set X and $x \in X$

then $|Gx| = \dfrac{|G|}{|\mathrm{Stab}(x)|}$

Proof:

have a map

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & Gx \\ g & \longmapsto & gx \end{array}$$

surjective.

given $y \in Gx$, ask: what is $\varphi^{-1}(y)$ $= h\mathrm{Stab}(x)$

answer: $\{g \in G \mid gx = y\}$

if $y = hx$ some h. $\quad = \{g \in G \mid gx = hx\}$

$= \{g \in G \mid h^{-1}gx = x\}$

$|h\mathrm{Stab}(x)| = |\mathrm{Stab}(x)|$

$= \{g \in G \mid h^{-1}g \in \mathrm{Stab}(x)\}$

$h\mathrm{Stab}(x) = \{g \in G \mid g \in h\mathrm{Stab}(x)\} = \{g \in G \mid h^{-1}g \in \mathrm{Stab}(x)\}$

$\quad\quad // \quad\quad\quad\quad ||$

$$h^{-1}g = s$$
$$hh^{-1}g = hs$$
$$g = hs$$

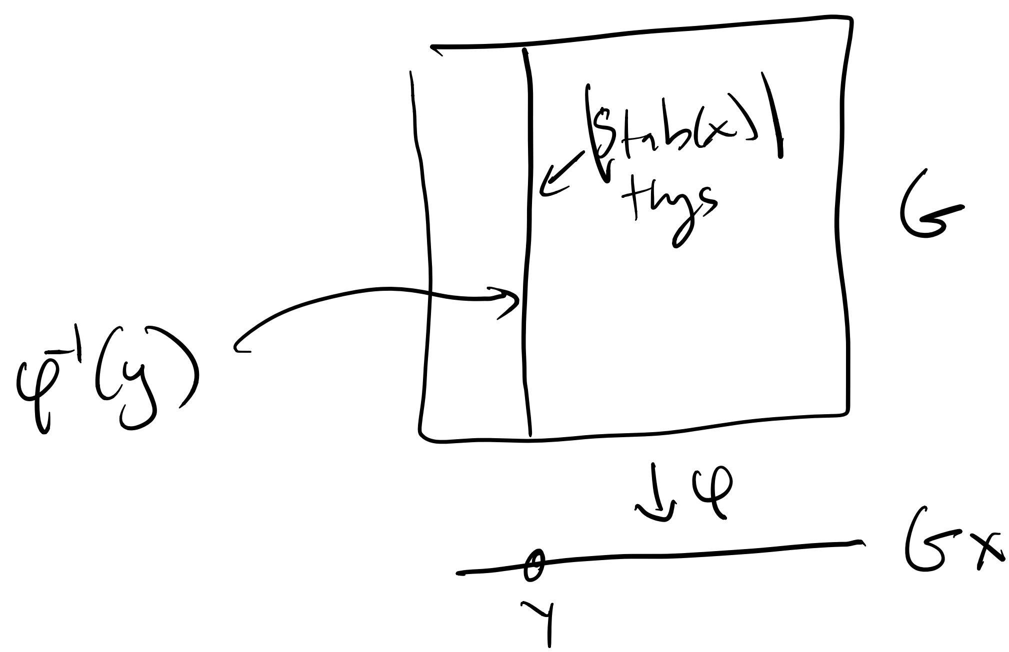
$$= \{g \in G \mid h^{-1}g = s \text{ some } s \in \mathrm{Stab}(x)\}$$

$$\{g \in G \mid g = hs \text{ for some } s \in \mathrm{Stab}(x)\}$$

$$G \xrightarrow{\varphi} Gx$$

if $y \in Gx$, $|\varphi^{-1}(y)| = |\mathrm{Stab}(x)|$

$$\Rightarrow |G| = |Gx| \cdot |\mathrm{Stab}(x)|$$



Meta fact of counting (def of multiplication)

if $\varphi : X \to Y$ is surjective

and if $n = |Y|$, $m = |\varphi^{-1}(y)|$ all $y \in Y$

then $|X| = nm$

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meaning of symbol

$$\varphi^{-1}(y) = \{x \in X \mid \varphi(x) = y\}$$

$$\parallel$$

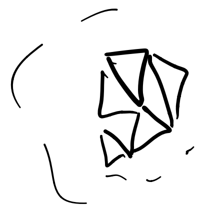
$$\varphi^{-1}(\{y\})$$

Example:

physical (rotational)

Count # of symmetries of a icosahedron

D20



G = symmetries

X = triangles (centers of Δ's)

$\overset{1}{{}_{3}\Delta_{2}}$

τ = rotate 120

τ^2 = rot. 240 = −120

e = do nothing C_3

3 symmetries

X = single orbit. $\Rightarrow$ $\underset{20}{\overset{\shortparallel}{|X|}} = \dfrac{|G|}{|\mathrm{Stab}(x)|}$

3

any particular Δ.

$$\boxed{|G| = 60}$$

Buckyball: 60 sides 20 hexagons,
12 pentagons

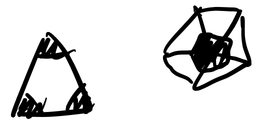
G symm, act on hexagons X = hex's.

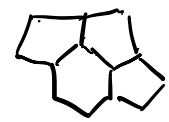
$$|X| = \frac{|G|}{|\mathrm{Stab}(x)|}$$

$\shortparallel$

20

any hexagon





need to rotate $120 = 2 \cdot 60$

to get a Bucky symmetry

Alternate translation of actions

Given an action $\hat{\alpha}$ of G on a set X

then we get a homomorphism

$$G \xrightarrow{\varphi_\alpha} S_X$$

via $\varphi_\alpha(g)(x) \equiv g \cdot x$

"α"

$$\varphi_\alpha(gh) \overset{?}{=} \varphi_\alpha(g)\,\varphi_\alpha(h)$$

$$\varphi_\alpha(gh)(x) = gh \cdot x$$

$$\overset{?}{=} \varphi_\alpha(g)\big(\underbrace{\varphi_\alpha(h)(x)}\big)$$

$$\varphi_\alpha(g)(\;h\cdot x\;)$$

$$g\cdot(h\cdot x)$$

✓

why is $\varphi_\alpha \in S_X$ $\quad \varphi_\alpha(g^{-1})\varphi_\alpha(g) = \varphi_\alpha(e)$

$\varphi_\alpha(e)(x) = e \cdot x$
$= x$

$\varphi_\alpha(g^{-1}) \circ \varphi_\alpha(g) = id_X$

$\Rightarrow \varphi_\alpha(g)$ is bijective (since it has an inverse fcn)

$\varphi_\alpha(g) \circ \varphi_\alpha(g^{-1}) = id_X$

Conversely, if $\varphi : G \longrightarrow S_X$ is any homomorphism,

get an action α_φ via

$$g \cdot x \equiv \varphi(g)(x)$$

<u>Claim:</u> this gives a bijection between

$$\{\text{actions of } G \text{ on } X\} \longleftrightarrow \{\text{hom's. } G \to S_X\}$$

$$\alpha \longmapsto \varphi_\alpha$$

$$\alpha_\varphi \longleftarrow\!\shortmid \varphi$$

Def An action is called "faithful" if it has no kernel. (I mean kernel = $\{e\}$)

Def If $\varphi : G \longrightarrow H$ is any homomorphism,

$$\ker \varphi = \{ g \in G \mid \varphi(g) = e \}$$

Remark if α is an action, $\boxed{\ker(\alpha) = \ker(\varphi_\alpha)}$

Lem if $\varphi : G \longrightarrow H$ is a hom, then $\ker \varphi < G$

Pf: $e \in \ker \varphi$ since $\varphi(e) = e$

if $g, h \in \ker \varphi$ then $\varphi(gh) = \varphi(g)\varphi(h)$
$= e \cdot e = e$

if $g \in \ker \varphi$ then $\varphi(g^{-1}) = \varphi(g)^{-1} = e^{-1} = e$
$\Rightarrow g^{-1} \in \ker \varphi$.

Def $H < G$ is <u>normal</u> if $\forall g \in G$, $gH = Hg$

<u>equivalently</u>: $h \in H, g \in G \Rightarrow ghg^{-1} \in H$

Pf: if $gH = Hg$ and $h \in H$,
all $g \in G$

ghg^{-1}

$gh \in gH = Hg$

$\Rightarrow gh = h'g$
same $h' \in H$

$\Rightarrow h'gg^{-1} = h' \in H$

conversely, if $ghg^{-1} \in H$ all $g \in G, h \in H$

$gH = \{gh \mid h \in H\} \subset \{h'g \mid h' \in H\}$ $=$ Hg

$ghg^{-1} \in H$

$ghg^{-1} = h'$

$gh = h'g$

$gH \subset Hg$, $Hg \subset gH$ similarly

so $=$.

__motivating example__

$g \in G$

$$\mathbb{Z} \xrightarrow{\varphi} \langle g \rangle \simeq C_m$$

$$n \longmapsto g^n$$

$$\underset{\substack{\| \\ \ker \varphi}}{\{n \in \mathbb{Z} \mid g^n = e\}} = m\mathbb{Z}$$

same m
"order"

$$\begin{tikzcd}
 & & \langle g \rangle \\
\mathbb{Z} \arrow[r, phantom, "\ni"] & i \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & g^i \\
 & i+m &
\end{tikzcd}$$

$$g^{i+m} = g^i$$

general principle

if $\varphi : G \longrightarrow \overline{G}$ is a surjective homom of gps

then can identify $\overline{G}$ w/ $G/\sim$ where $g \sim g'$ if differ by $\ker\varphi$

$g \sim g'$ if $gn = g'$ some $n \in \ker \varphi$

$\Updownarrow$

$n'g = g'$ some $n' \in \ker \varphi$