Suppose G is a group, $N \trianglelefteq G$ (normal subgroup)

then we can define a new group G/N (yesterday $G/\sim$)

elements in G/N are coset $gN = [g]$

w/ group operation $[g][h] = [gh]$

$(gN) \cdot (hN) = ghN$

<u>Notation</u> "Set notation"

$S, T \subset G$, set $ST = \{st \mid s \in S, t \in T\}$

$gShTh^{-1} = \{gshth^{-1} \mid s \in S, t \in T\}$

$\boxed{gNhN = \{gnhn' \mid n, n' \in N\}}$

$gHg^{-1} = \{ghg^{-1} \mid h \in H\}$

<u>Exercise</u>: show that if $N \trianglelefteq G$, $gNhN = ghN$

$\supset$

$ghn = gehn \in gNhN$

$\subset$

$gnhn = ghn'n \in ghN$

$Nh = hN$

$nh \in hN$

$nh = hn'$

ex: $G = \mathbb{Z}$ $N = 3\mathbb{Z}$

Side comment if G is Abelian (commutative)

then $H < G \Rightarrow H \lhd G$

because $gH = \{gh \mid h \in H\} = \{hg \mid h \in H\}$

$= Hg$

$3\mathbb{Z} \lhd \mathbb{Z}$

elements of $\mathbb{Z}/3\mathbb{Z} = \{n + 3\mathbb{Z} \mid n \in \mathbb{Z}\}$

$(gN)+(hN)$ $\quad gNhN$

$(n + 3\mathbb{Z}) + (m + 3\mathbb{Z}) = \{n + 3z_1 + m + 3z_2\}$

$ghN \quad \{n + m + 3z\}$

$=$

$n + m + 3\mathbb{Z}$ $\quad (3\mathbb{Z} + 3\mathbb{Z} = 3\mathbb{Z})$

$n = 7$ $\quad \{7 + 3z \mid z \in \mathbb{Z}\}$

$= 7 + 3\mathbb{Z}$ $\quad = \{7, 10, 4, \boxed{1}, -2, 13\}$

$\bar{1} =$ $\quad 0, 1, 2$

$\bar{0}, \bar{1}, \bar{2}$

<u>Recall</u> Quotients "are" surjective maps

Set theoretically:

if $f: X \longrightarrow Y$ surjective of sets

then $\exists!$ equivalence relation $\sim$ on X

such that f factors as

$$f = \bar{f} \circ \pi$$

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Y \\
& X/\sim \arrow[ur, "\bar{f}", "\sim"'] &
\end{tikzcd}$$

bijective

surjective = onto
"
epic

injective = 1-1
"
monic

$x \sim x' \iff f(x) = f(x')$

Y = labels on buckets

f: bucket chooser.

"universal property of equiv relations"

Let X be a set, $\sim$ an eq. rel. then if $f: X \to Y$

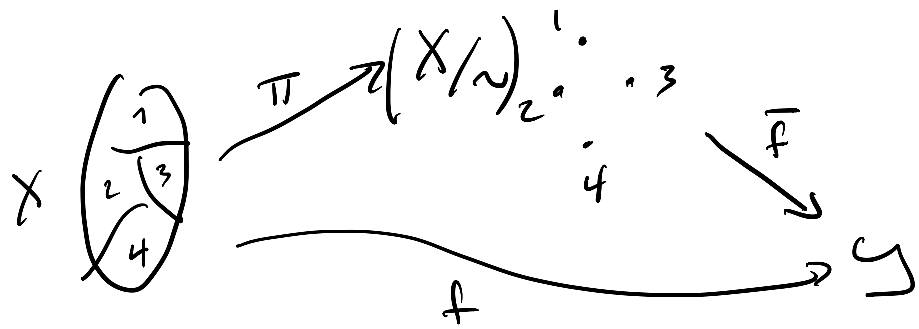
then $\exists\, \bar{f}: X/\sim \longrightarrow Y$ such that f factors

as

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & Y \\
& X/\sim \arrow[ur] &
\end{tikzcd}$$

if and only if $x \sim x' \Rightarrow f(x) = f(x')$

and if $\bar{f}$ exists like this, it is <u>unique</u>.



$$\begin{tikzcd} & (X/\sim) \bullet ab \\ X = \{a, b\} \arrow[ur] \arrow[rr, "id"'] & & Y = \{a, b\} \end{tikzcd}$$

$a \sim b$

"universal prop of ~ rels"

in other words

$$Maps(X/\sim, Y) = \{ f : X \to Y \mid x \sim x' \Rightarrow f(x) = f(x') \}$$

are in bijection

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if $f : G \to H$ is a surjective map

then $\exists \sim$ rel on G (as a set) s.t. map factors

as

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] & H \\ G/\sim \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

homomorphism

bijection

inherits a grp structure from H

$f = \bar{f} \circ \pi$

$\bar{f}^{-1} \circ f = \cancel{\bar{f}^{-1} \circ \bar{f}} \circ \pi$

$\bar{f}^{-1} \circ f = \pi$

a homomorphism

$\sim$ given by $N \trianglelefteq G$

on $G/\sim \; = G/N$

C_3

$$\begin{array}{ccc} 1 & \sigma & \sigma^2 \\ \downarrow & \downarrow & \downarrow \\ x & y & z \end{array}$$

$xx = x \qquad y^2 = z$

<u>we've proved:</u>

if $f : G \to H$ is an epic homomorphism
epimorphism

then $\exists\, N \trianglelefteq G$ such that f factors as

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "\pi"'] & & H \\ & G/N \arrow[ur] & \end{tikzcd}$$

isomorphism

$g \mapsto gN$

$N \mapsto e$

$$\boxed{\text{in fact: } N = \ker f}$$

Claim $N = \ker f$.

$\subset$

$n \in N \Rightarrow n \in \ker f$

$$\begin{tikzcd} G \arrow[rr] \arrow[dr] & & H \\ & G/N \arrow[ur] & \end{tikzcd}$$

$n \mapsto nN = N \mapsto e$

$[n]$

conversely if

$g \in \ker f \Rightarrow g \mapsto e$, $g \mapsto gN \mapsto e$

surjective

$\Downarrow$

$gN = N$

$[g] = \underset{N}{\underset{\shortparallel}{[e]}}$

$g \in N$

Our <u>First Isom. thm</u>

if $f: G \longrightarrow H$ is surjective hom

then $H \cong G/\ker f$

Note if $X \xrightarrow{f} Y$ set map then f can be factored as

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \\ & Y' \arrow[ur] & \end{tikzcd}$$

where $Y' \to Y$ injective (inclusion)

and $X \to Y'$ surjective.

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \\ & \operatorname{im} f \arrow[ur, hook] & \end{tikzcd}$$

Apply to groups:

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "f'"'] & & H \\ & G' = \operatorname{im} f \arrow[ur, hook, "i"'] & \end{tikzcd}$$

inclusion

check: $\bar{f}$, i are also hom's.

$$\bar{f}(gh) = f(gh) = f(g)f(h) = \bar{f}(g)\bar{f}(h)$$

$f : \mathbb{R} \to \mathbb{R}$

$\bar{f} : \mathbb{R} \to \{ r \in \mathbb{R} \mid r > 0 \}$

$f : X \to Y$ $\qquad \operatorname{im} f = Y' \subset Y$

$f \subset X \times Y$ s.t.

$= \{ (x, y) \; - - - \}$

Our theorem of the day

If $f: G \longrightarrow H$ any homomorphism $N = \ker f$

then f factors as

$$\begin{tikzcd} G \arrow[r] \arrow[rr, bend left, "f"] & G/N \arrow[r, "\sim"] & (\operatorname{im} f) \hookrightarrow H \end{tikzcd}$$

quotient   iso   subgroup

Cor: $G/\ker f \simeq \operatorname{im} f$ "1st iso. theorem"

If X a set $\sim$ equiv rel, Y set

$$\operatorname{Maps}(X/\sim, Y) \xrightarrow{\alpha} \operatorname{Map}(X, Y)$$

$$(\bar{f} : X/\sim \to Y) \longmapsto (X \to X/\sim \xrightarrow[\bar{f}]{} Y)$$

then α gives a bijection between $\operatorname{Maps}(X/\sim, Y)$ & the subset $\{f : X \to Y \mid x \sim x' \Rightarrow f(x) = f(x')\}$ of $\operatorname{Maps}(X, Y)$