# Recap

Last time: $f: X \longrightarrow Y$ $\underbrace{\text{map of sets}}_{=\text{function}}$,

then we can factor $f$ as

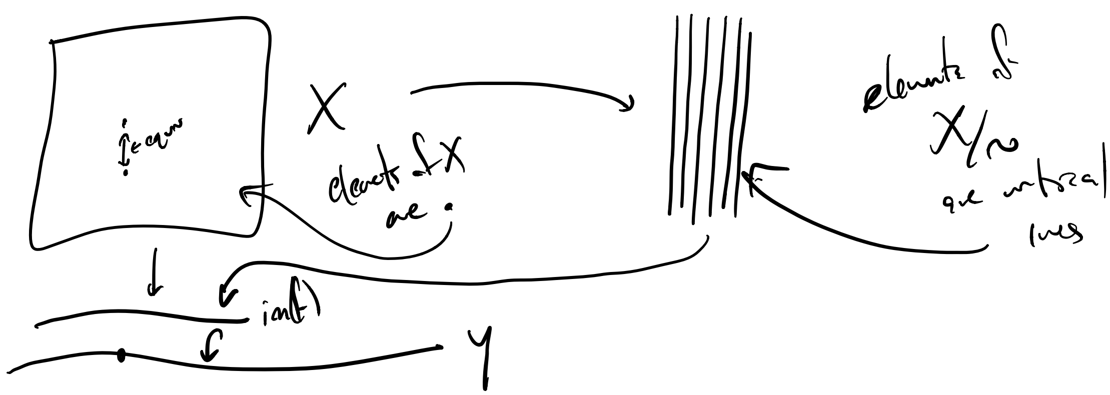
$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[d, two heads] & & Y \\
X/\sim \arrow[r, "\sim"] & \mathrm{im}(f) \arrow[ur, hook] &
\end{tikzcd}$$

surjective

bijective

injection (inclusion)

$\sim$ defined by $x_1 \sim x_2 \iff f(x_1) = f(x_2)$



$f: G \longrightarrow H$ group hom.

$$\begin{tikzcd}
G \arrow[rrr, bend left, "f"] \arrow[dr] \arrow[drr, bend right, "\text{is a group hom.}"'] & & & H \\
& G/\sim \arrow[r, "\sim"] & \operatorname{im}(f) \arrow[ur] & 
\end{tikzcd}$$

$\leftarrow$ is a subgp of $H$

$G = \{1,-1\} < \mathbb{R}^{\times}$

$H = \mathbb{R}^{\times}$

$f: \{1,-1\} \longrightarrow \mathbb{R}^{\times}$

$\operatorname{im}(f) = G < H$

$G/\sim \; = G/N \Leftarrow$

$$\boxed{\begin{array}{l} \underline{\text{Claim:}} \text{ if } f: G \longrightarrow H \text{ is a hom then } \operatorname{im}(f) < H \\ \text{and the map } G \longrightarrow \operatorname{im}(f) \text{ ``}f\text{''} \\ \text{is a group homomorphism.} \end{array}}$$

If $G$ a gp $X$ a set

and $\varphi: G \longrightarrow X$ is a bijection

define $x \cdot y = \varphi((\varphi^{-1}x)(\varphi^{-1}y))$

then $X$ is a gp and $\varphi$ an isomorphism.

<u>Summary</u>

If $G \xrightarrow{f} H$ is a gp. homomorphism, $N = \ker f$

then $G/N \simeq \operatorname{im}(f)$

and in particular if f is surjective then

$$G/N \cong H$$

1st isom. theorem.

<u>Ex</u>: consider parity

$$\varphi : \mathbb{Z} \longrightarrow \{\text{even}, \text{odd}\}$$

e o

def of gp structure.

$$\varphi(n) = \begin{cases} e & \text{if } n \text{ even} \\ o & \text{if } n \text{ odd} \end{cases}$$

$e + e = e$

$e + o = o$

$o + o = e$

$o + e = o$

$\varphi(n) + \varphi(m)$

$$\overset{\shortparallel}{\varphi(n+m)} = \begin{cases} n, m \text{ even}, & \text{even} \Rightarrow e \\ n \text{ even } m \text{ odd}, & \text{odd} \\ n \text{ odd } m \text{ even}, & \text{odd} \\ n, m \text{ odd} & e \end{cases}$$

$\ker \varphi = \text{even \#s} \Rightarrow (\text{even \#s}) \trianglelefteq \mathbb{Z}$

$$\mathbb{Z} / \underbrace{(\text{even\#s})}_{2\mathbb{Z}} = \{e, o\}$$

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Still in chapter 2:

Correspondence theorem

## Correspondence theorem (Artin)

if $f: G \longrightarrow H$ is a surjective homomorphism with $\ker f = N$ then there is a bijective correspondence between subgroups of $H$ & subgroups of $G$ which contain $N$.

$$\left\{\begin{matrix}\text{subgroup of } G \\ \text{containing } N\end{matrix}\right\} \qquad\qquad \left\{\begin{matrix}\text{subgroups} \\ \text{of } H\end{matrix}\right\}$$

$$K \longmapsto f(K) = \{f(k) \mid k \in K\}$$

$K < G$

$N \subset K$

$$f^{-1}(L) \longleftarrow\!\shortmid L < H$$

$$f^{-1}(L) = \{g \in G \mid f(g) \in L\}$$

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## Counting Interlude

recall Lagrange: we showed

$G$ is partitioned into left cosets for any subgp $H < G$.

$G$ = disj. union of some $gH$'s. various $g$'s.

and these all have the same size $= |H|$

$|G| = |H| \cdot \underbrace{\# \text{cosets}}_{[G:H]}$

<u>Def</u> $[G:H] = \#$ cosets of $H$ in $G$.

<u>Note:</u> if $N \lhd G$, consider $G/N$ elements are cosets.

$\Rightarrow [G:H] = |G/N|$

<u>ex:</u> $2\mathbb{Z} < \mathbb{Z}$ has index 2.

$\mathbb{Z}/2\mathbb{Z} \xrightarrow{\sim} C_2$

$0 + 2\mathbb{Z}$

$1 + 2\mathbb{Z}$

$|G| = |N| \cdot [G:N]$

$= |N| \, |G/N|$

so if $G \xrightarrow{f} H$ surjective

$G/\ker f \simeq H$ $\quad |G/\ker| \,\big|\, |G|$

$\Rightarrow |H| \,\big|\, |G|$

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if $f: G \to H$

$n \in \ker f$ $\quad g \in G$

$gng^{-1} \in \ker f$

## Examples

Symmetries of geometric figures.

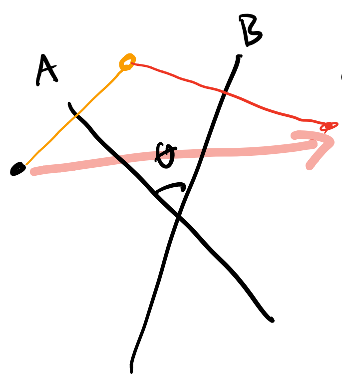
A plane symmetry = a distance preserving function from the plane to itself

$\parallel$

$\mathbb{R}^2$

= combinations of reflections, rotations, translations.

turns out

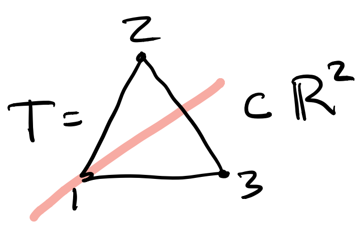



$($ refl A $\circ$ refl B $)$ = rotate $2\theta$

$\parallel$


translation by 2·dist A to B

## Symmetries of triangle (equilateral)




- rotate $120°$
- do nothing.
- reflect along line

want to find ~~motions~~ symmetries of the plane $\varphi: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that take $T$ to itself how many symmetries?

6

<u>orbit stabilizer:</u> let $G$ = symmetries of $T$.

$G$ acts on the set of vertices

$G \times \{1,2,3\} \to \{1,2,3\}$

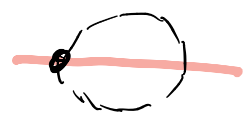
orbit of $1 = \{1,2,3\}$

stabilizer of $1 = \{e, \text{reflect through } 1\}$

<u>orbit stab:</u> $|G| = |\text{orbit}||\text{stab}|$

$= 3 \cdot 2 = 6.$

6 symmetries of $\Delta$.

if we consider a regular $n$-gon 

# symm. of $n$-gon? $2n$

$G$ acts on vertices $|\text{orbit}| = n$

$|\text{stab}| = 2$

<u>Def</u> $D_n$ "dihedral group" is the gp of symmetries of a regular $n$-gon.

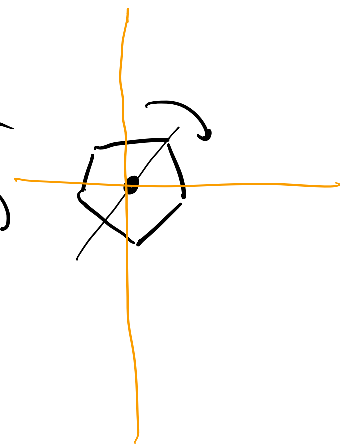
$|D_n| = 2n$

structure of $D_n$

elements are all lnr transformations of plane

because all fix the origin

(same distances are preserved & middle is the unique pt equidistant to all vertices)



orthogonal lnr transformations

"

preserve length.

$$O_2(\mathbb{R}) \subset GL_2(\mathbb{R})$$

"

$$\{T \in M_2(\mathbb{R}) \mid TT^t = I_2\}$$

$$D_{2n} \longrightarrow \pm 1$$