Symmetries

We defined : A symmetry (isometry) of the plane is a distance preserving map $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

Def An isometry $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a distance preserving function.

Ex: if $v \in \mathbb{R}^n$ vector $t_v(a) = a + v$ is an isometry.

Def if $a, b \in \mathbb{R}^n$ $d(a,b) = \|a-b\| = \sqrt{\langle a-b, a-b \rangle}$

$\|v\| = \sqrt{\langle v,v \rangle}$ $\langle v, w \rangle = v^t w$

$$d(t_v(a), t_v(b)) = d(a,b)$$

$$d(a+v, b+v) = \| a+v - (b+v) \| = \|a-b\|$$

Ex: if $T \in O_n(\mathbb{R}) = \{ L \in M_n(\mathbb{R}) \mid LL^t = I_n \}$

then $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an isometry

since $\langle Tv, Tw \rangle = (Tv)^t Tw = v^t T^t T w$

$= v^t I_n w$

$= v^t w = \langle v, w \rangle$

$$\Rightarrow d(Tv, Tw) = \| Tv - Tw \|$$

$$= \| T(v-w) \| = \langle T(v-w), T(v-w) \rangle$$

$$= \langle (v-w), v-w \rangle$$

$$= \dots = d(v,w)$$

Theorem If $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an isometry then $f = t_v \circ T$ for $T \in O_n(\mathbb{R})$, $v \in \mathbb{R}^n$ for a unique T & v.

Lemma If $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an isometry s.t. $f(0) = 0$ then $f = T \in O_n(\mathbb{R})$

Pf. of thm assuming lemma

if f general isometry, consider $g = t_{-f(0)} f$

$$g(0) = (t_{-f(0)} \circ f)(0) = t_{-f(0)}(f(0)) = f(0) - f(0) = 0$$

so g is an origin preserving isom. $\xRightarrow{(\text{Lem})}$ $g = T \in O_n(\mathbb{R})$

$$\Rightarrow T = t_{-f(0)} f$$

$$t_{f(0)} \circ T = t_{f(0)} t_{-f(0)} f = f$$

uniqueness:

if $t_a T = t_b S$ want to show $a = b$, $T = S$.

apply 0 to both sides

$$t_a T(0) = t_b S(0)$$

$$t_a(0) = t_b(0) \Rightarrow a = b$$

and $t_a T = t_a S$

$$t_{-a} t_a T = t_{-a} t_a S \Rightarrow T = S$$

Notice:

Isometries of $\mathbb{R}^n$ for a group (composition of isometries is another, inverses

$$(t_a T)^{-1} = T^{-1} t_{-a}$$

$$t_a T T^{-1} t_{-a} = t_a t_{-a} = id \ )$$

To prove lemma, need to show: if $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $f(0) = 0$ an isometry then f is a linear operator, preserving dot products.

Sublemma: if $x, y \in \mathbb{R}^n$ and $\langle x, x\rangle = \langle x, y\rangle = \langle y, y\rangle$ then $x = y$.

Pf: $\|x - y\| = \langle x-y, x-y\rangle$

$$= \langle x, x\rangle - \langle y, x\rangle - \langle x, y\rangle + \langle y, y\rangle$$

$$= 0 \Rightarrow x - y = 0 \Rightarrow x = y.$$

Pf of lemma

Suppose f is an isometry, $f(0)=0$

and $d(x,y) = d(f(x),f(y))$

$$\|f(x)-f(y)\|^2 = \langle f(x)-f(y), f(x)-f(y)\rangle$$

$$= \cancel{\langle f(x),f(x)\rangle} - 2\langle f(x),f(y)\rangle + \cancel{\langle f(y),f(y)\rangle}$$

$$\|x-y\|^2$$

$$= \cancel{\langle x,x\rangle} - 2\langle x,y\rangle + \cancel{\langle y,y\rangle}$$

but $\langle x,x\rangle = \|x\|^2 = d(x,0) = d(f(x),0)$

$= \|f(x)\|^2 = \langle f(x),f(x)\rangle$

$$\Rightarrow \quad \langle f(x),f(y)\rangle = \langle x,y\rangle$$

$\Rightarrow f$ preserves $\langle\ ,\ \rangle$.

Linear transformation?

$$f(x+y) = f(x)+f(y)\ ?$$

ETS: $\langle f(x+y), f(x+y)\rangle \overset{\checkmark}{=} \langle f(x+y), f(x)+f(y)\rangle$

$\overset{\checkmark}{=} \langle f(x)+f(y), f(x)+f(y)\rangle$

$$\langle f(x+y), f(x)+f(y)\rangle = \langle f(x+y), f(x)\rangle + \langle f(x+y), f(y)\rangle$$

$$= \langle x+y, x\rangle + \langle x+y, y\rangle$$

$$= \langle x+y, x+y\rangle$$

$$= \langle f(x+y), f(x+y)\rangle$$

$$\langle f(x)+f(y), f(x)+f(y)\rangle$$

$$= \langle f(x), f(x)\rangle + 2\langle f(x), f(y)\rangle + \langle f(y), f(y)\rangle$$

$$= \langle x, y\rangle + 2\langle x, y\rangle + \langle y, y\rangle$$

$$= \langle x+y, x+y\rangle = \langle f(x+y), f(x+y)\rangle$$

and $f(\lambda x) \stackrel{?}{=} \lambda f(x)$

$$\langle f(\lambda x), \lambda f(x)\rangle = \lambda\langle f(\lambda x), f(x)\rangle$$

$$= \lambda\langle \lambda x, x\rangle$$

$$\langle \lambda x, \lambda x\rangle = \lambda^2\langle x, x\rangle$$

$$= \lambda^2\langle f(x), f(x)\rangle$$

$\shortparallel$

$\langle f(\lambda x), f(\lambda x)\rangle = \langle \lambda f(x), \lambda f(x)\rangle$

Def $\mathrm{Isom}(\mathbb{R}^n)$ (AKA Euclidean group)

= group of isometries of $\mathbb{R}^n$, operation is composition.

$\mathbb{R}^n < \mathrm{Isom}(\mathbb{R}^n)$

$a \longmapsto t_a$

Exercise show that $\mathbb{R}^n \lhd \mathrm{Isom}(\mathbb{R}^n)$

$$(t_a T)^{-1} = T^{-1} t_{-a}$$

$$\mathbb{R}^n < \mathrm{Isom}(\mathbb{R}^n)$$

$$a \longleftrightarrow t_a$$

want to show, if $g \in \mathrm{Isom}(\mathbb{R}^n)$

$g = t_c T$ $\qquad g t_a g^{-1} \in \mathbb{R}^n$

" t_b some b.

$$\underbrace{t_c T t_a T^{-1} t_{-c}}_{h} \overset{?}{=} t_b \text{ some } b.$$

✓ $b = Ta$.

$$t_c\left(T\left(t_a\left(T^{-1}\left(t_{-c}(v)\right)\right)\right)\right) = h(v)$$

$$t_{-c}(v) = v - c$$

$$T^{-1}(v-c) = T^{-1}v - T^{-1}c$$

$$t_a\left(T^{-1}\left(t_{-c}(v)\right)\right) = T^{-1}v - T^{-1}c + a$$

$$T\left(T^{-1}v - T^{-1}c + a\right) = v - c + Ta$$

$$t_c(v - c + Ta) = v + Ta$$

$$h(v) = v + Ta = t_{Ta}(v)$$

$$\Rightarrow h = t_{Ta}$$

$$\Rightarrow$$

$$\mathbb{R}^n \lhd \mathrm{Isom}(\mathbb{R}^n)$$

$$\begin{aligned} \mathrm{Isom}(\mathbb{R}^n) &\longrightarrow O_n(\mathbb{R}) \\ t_a T &\longmapsto T \end{aligned}$$