Advanced Calculus II, Fall 2022, Homework 2

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Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. (required) Consider the set $S = \{1/n | n \in \mathbb{Z}_{>0}\}$ and let $T = \mathbb{R} \setminus S$. Show that every point in T is in the interior of T except for 0.

2. (required) Suppose X is a complete metric space and $S \subset X$ is closed. Show that S is also complete (considered as a metric space itself).

3. (required) Let X be a metric space and $x \in X$ an element. Show that $\{x\}$ is closed.

4. (required) Show that there is no metric preserving map $\phi : \mathbb{R}^2 \to \mathbb{R}$ (using the standard Euclidean metric). That is, there is no function ϕ such that $d(\phi(x), \phi(y)) = d(x, y)$ for all x, y.

5. (required)

If X is a metric space and $S \subset X$ is a subset, we define the closure of S, denoted $\overline{S}$, to be the points $x \in X$ such that for every $\epsilon > 0$, we have $B_\epsilon(x) \cap S \neq \emptyset$.

(a) Show that if $x \in \overline{S}$ then there exists a sequence (s_n) with $s_n \in S$ with $\lim_{n \to \infty} s_n = x$.

(b) Show that if there exists a sequence (s_n) with $s_n \in S$ with $\lim_{n \to \infty} s_n = x$, then $x \in \overline{S}$.

(c) Show that $\overline{S}$ is closed.

6. (optional) Find an example of a metric space (X, d), points $x, y \in X$ and a real number $r > 0$ such that $d(x, y) = r$ but $y \notin \overline{B_r(x)}$ (the closure of the ball (see the previous problem)).

7. (optional) Recall that if we are given a metric space (X, d), we can define a notion of an open set with respect to that metric. Suppose we are given two different metrics d' and d on X such that there exists a $\lambda > 0$ such that for all $x, y \in X$, we have $d(x, y) \leq \lambda d'(x, y)$. Show that if U is open with respect to the metric d', it is also open with respect to the metric d.

8. (optional) Let (X, d) be a metric space and consider the new metric d' defined by $d'(x, y) = \min\{d(x, y), 1\}$. Show that a subset $S \subset X$ is open with respect to d if and only if it is open with respect to d'.

9. (optional) Suppose X is a metric space such that the distance function d only takes on finitely many values. Show that in this case X is complete.

10. (optional) Consider the rational numbers $X = \mathbb{Q}$ as a metric space (regarded as a subspace of $\mathbb{R}$). Let $S = \{r \in \mathbb{Q} \mid \sqrt{2} < r < \sqrt{2}\}$. Show that S is closed in X.

11. (optional) Suppose X is a metric space and $S \subset X$ is a subset such that every sequence (a_n) with $a_n \in S$ converges to some element of X (not necessarily in S!). Show that $\overline{S}$ is sequentially compact: every sequence (b_n) with $b_n \in \overline{S}$ converges to some element of $\overline{S}$.