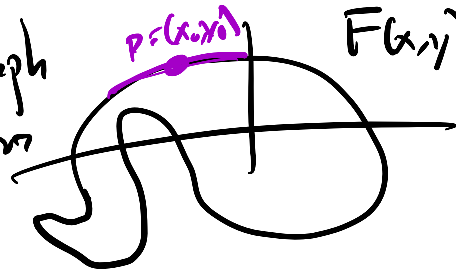
Goal: Differentiation & Implicit function theorem.

Basic Idea:

We may have a graph which is not a fn of fn



$F(x,y)=0$

$x^2+y^2=1$

Implicit fcn thm:

locally, as long as we have a nonvanishing derivative $\frac{\partial F}{\partial x}$ or $\frac{\partial F}{\partial y}$

can write either y as fcn of x or vise-versa.

$\frac{\partial F}{\partial x}(x_0,y_0)\neq 0$

Want: $F_1(\vec{x}) = 0,\ F_2(\vec{x}) = 0 \ \ldots\ldots,\ F_m(\vec{x}) = 0$

$\vec{x}\in\mathbb{R}^{n+m}$

Flash review of linear algebra

Recall:

vectors, scalars

A vector space V over a field F $(F = \mathbb{R}$ or $\mathbb{C})$

is a set w/ operations

$$+ : V \times V \longrightarrow V \qquad \text{and } 0 \in V$$
$$x, y \longmapsto x + y \qquad \text{distinguished element}$$

$$\cdot : F \times V \longrightarrow V$$
$$\lambda, x \longmapsto \lambda x$$

Such that:

Additive structure

- $x + (y + z) = (x + y) + z \quad (= x + y + z)$
- $x + y = y + x$
- $\forall x \in V, \ x + 0 = x$
- $\forall x \in V, \exists y \in V$ s.t. $x + y = 0$

  we call this y (which is unique) $y = -x$

Scalar structure

- $\lambda(x + y) = \lambda x + \lambda y$

$\bullet\ (\lambda+\mu)x = \lambda x + \mu x$

If V is a vectr space over F, $W \subset V$ is a subset, we say W is a subspace if $W \neq \emptyset$

$x, y \in W \Rightarrow x+y \in W$

$x \in W, \lambda \in F \Rightarrow \lambda x \in W$

In this case, operations turn W into an F vectr space.

(All of this is done in beginning of ch 8)

<u>examples</u>

- $\mathbb{R}^n$ is an $\mathbb{R}$-vectr space

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1+y_1, \dots, x_n+y_n)$$

$$\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$$

- F^n • $\mathbb{C}^n$

- Continuous functions $\mathbb{R} \to \mathbb{R}$ $\quad C(\mathbb{R}, \mathbb{R})$

$$(f+g)(x) = f(x) + g(x)$$
$$(\lambda f)(x) = \lambda f(x)$$

- Arbitrary functions $X \to \mathbb{R}$ $\quad \mathrm{Fun}(X, \mathbb{R})$
- X is a metric space, $C(X, \mathbb{R})$ is a subspace of $\mathrm{Fun}(X, \mathbb{R})$
- If V is any vector space, X any set, $\mathrm{Fun}(X, V)$ is a vector space!

$$(f+g)(x) = f(x) + g(x) \in V$$
$$(\lambda f)(x) = \lambda f(x)$$

<u>**Def**</u> if X, Y are vector spaces over a field F, we say that a map $\varphi: X \longrightarrow Y$ is a linear transformation if

$$\varphi(x_1 + x_2) = \varphi(x_1) + \varphi(x_2) \;\; \& \;\; \varphi(\lambda x) = \lambda \varphi(x).$$

these are also called <u>homomorphisms</u>

Def $\mathrm{Hom}_F(X,Y) = \{\text{lin. trans. } \varphi : X \to Y\}$

Notice: $\mathrm{Hom}_F(X,Y) \subset \mathrm{Fun}(X,Y)$

$\mathrm{Hom}_F(X,Y)$ " a subspace. $\mathrm{Fun}(X,Y)$ " a vector space.

ex: if $\varphi, \psi \in \mathrm{Hom}_F(X,Y)$ then

$\varphi + \psi \in \mathrm{Hom}_F(X,Y)$ as well !

e.g.: $(\varphi+\psi)(x_1+x_2) \overset{?}{=} (\varphi+\psi)(x_1) + (\varphi+\psi)(x_2)$

$$= \varphi(x_1+x_2) + \psi(x_1+x_2)$$

$$= \varphi(x_1)+\varphi(x_2) + \psi(x_1)+\psi(x_2)$$

Recall: $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m) = \mathrm{Mat}_{m,n}(\mathbb{R})$

is a subspace of $\mathrm{Fun}(\mathbb{R}^n, \mathbb{R}^m)$

$= \mathbb{R}^{nm}$ is a vector space as described

same vector space structures!

Recall: linr independence, spanning, basis, dimension.

If X, Y v-spcs over a field F, and

$\varphi: X \to Y$ is a lnr. trans which is bijective (1-1 & onto)

then $\varphi^{-1}: Y \to X$ exists and is also a lnr trans.

in this case, we say φ is an isomorphism

and that X & Y are isomorphic.

example

$$\mathbb{R}^2 \xrightarrow{\varphi} X = \{\text{lnr polynomials in } x\}$$

an isomorphism

$$(a,b) \longmapsto ax+b \qquad ax+b$$

$\mathbb{R}^2$ is isomorphic to X

$L_1 = ax+b$

$L_2 = cx+d$

$L_1 + L_2 = (a+c)x + (b+d)$

Pf: of isom:

φ 1-1

if $\varphi(a,b) = \varphi(c,d) \Rightarrow$
$ax+b = cx+d \to a=c,\ b=d$

φ ln. trans

φ onto

given L lnr. since L has form $ax+b$, $L = \varphi(a,b)$ ✓

Recall: if X is an n-dimensional v.space over F then X is isomorphic to F^n.

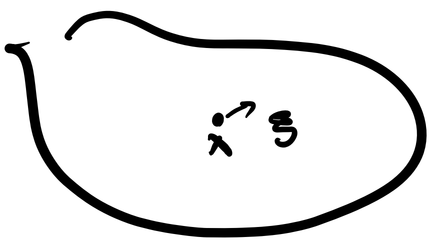
Goal: Given X, Y vector spaces, over $\mathbb{R}$

$f: X \to Y$ function, $x \in X$

$$\frac{\partial f}{\partial \text{"}\xi\text{"}}(x) = \lim_{h \to 0} \frac{f(x+h\xi) - f(x)}{\|h\xi\|}$$

$f(x)$

vect $\xi \in X$
vector



X

" Gateaux derivative "
