Operator norm

Recall: normed vector spaces

Given: $T: X \to Y$ lin. trans. between normed vector spaces,

Define $\|T\| = \sup \left\{ \frac{\|Tx\|}{\|x\|} \mid x \neq 0 \text{ in } X \right\}$

operator norm

note: this can be ∞!

we say T is **unbounded** if $\|T\| = \infty$

otherwise we say T is **bounded**.

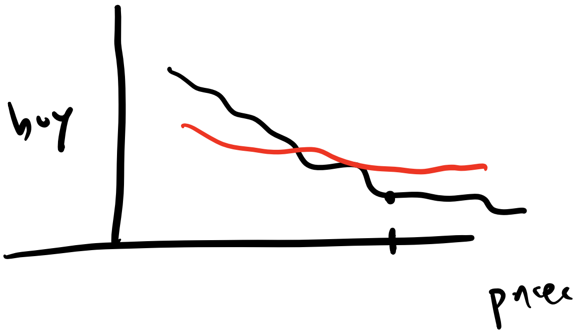
$B(X,Y) = \{ T \in L(X,Y) \mid T \text{ is bounded} \}$

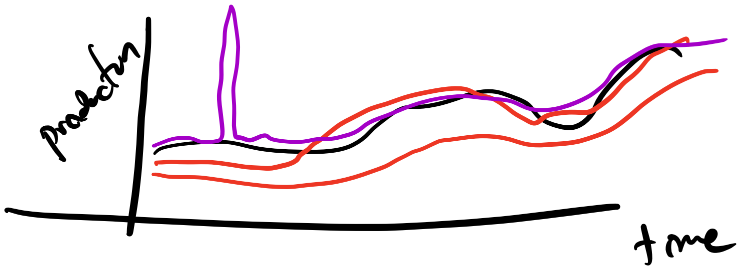
(**Notation:** $Tx = T(x)$)

Spoiler: if X finite dimensional $\Rightarrow$ any $T: X \to Y$ is **bounded**.

Spoiler: if $T: X \to Y$ is a linear transformation then it is continuous if and only if it's bounded!

Lemma: $\|T\| = \sup \{ \|Tx\| \mid x \in X \text{ s.t. } \|x\| = 1 \}$





distance uses rates of change "Sobolev metrics"

$$d(f,g) = \int_0^1 |f(x)-g(x)|\,dx + \int_0^1 |f'(x)-g'(x)|\,dx$$

<u>Example</u> $X = Y = \mathbb{R}$

$$T : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \lambda x$$

$$\|T\| = \sup\left\{ \frac{\|Tx\|}{\|x\|} \right\}$$

$$= \sup\left\{ \frac{\|\lambda x\|}{\|x\|} \right\}$$

$$\|\lambda x\| = \lambda \|x\|$$

$$= \sup\{\lambda\}$$

$$= \lambda$$

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$T \longleftrightarrow \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$ $\qquad Tx = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5x_1 \\ 2x_2 \end{bmatrix}$

$$\sup \left\{ \frac{\|Tx\|}{\|x\|} = \frac{\sqrt{(5x_1)^2 + (2x_2)^2}}{\sqrt{x_1^2 + x_2^2}} \right\}$$

$$= \sup \left\{ \sqrt{\frac{(5x_1)^2 + (2x_2)^2}{x_1^2 + x_2^2}} \right\}$$

$$= \sqrt{\sup \left\{ \frac{(5x_1)^2 + (2x_2)^2}{x_1^2 + x_2^2} \,\middle|\, (x_1, x_2) \neq \vec{0} \right\}}$$

guess $\sup \left\{ \frac{(5x_1)^2 + (2x_2)^2}{x_1^2 + x_2^2} \right\} = 25$

$x_1 = 1 \quad x_2 = 0$

$\Rightarrow 25 \leq \sup \{ \quad \}$

$$\frac{(5x_1)^2 + (2x_2)^2}{x_1^2 + x_2^2} \leq \frac{(5x_1)^2 + (2x_2)^2}{x_1^2 + x_2^2} + \frac{(\sqrt{21}x_2)^2}{x_1^2 + x_2^2}$$

$$= \frac{(5x_1)^2 + (5x_2)^2}{x_1^2 + x_2^2} = \frac{25(x_1^2 + x_2^2)}{x_1^2 + x_2^2}$$

$$= 25$$

$$\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|}\right\} = \sup\left\{\sqrt{\frac{(5x_1)^2+(2x_2)^2}{x_1^2+x_2^2}}\right\}$$

$$= \sqrt{\underbrace{\sup\left\{\frac{(5x_1)^2+(2x_2)^2}{x_1^2+x_2^2}\right\}}_{25}}$$

$$= \sqrt{25} = 5$$

<u>Natural generalization:</u> if

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ diagonal $\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

then $\|T\| = \max\{|\lambda_i|, i=1,\dots,n\}$

more generally, if T is diagonalizable over $\mathbb{R}$

then $\|T\| = \max\{|\lambda| \mid \lambda \text{ is an eigenvalue of } T\}$

<u>Some facts about operator norm:</u>

If X, Y normed vector spaces, $A, B \in B(X,Y)$
$c \in \mathbb{R}$

1) $\|A+B\| \le \|A\| + \|B\|$

2) $\|cA\| = c\|A\|$

3) $\|A\| = 0 \iff A = 0$

In particular: $B(X,Y)$ are a vector subspace of $L(X,Y)$ & it has a norm via operator norm.

if $A \in \mathbf{B}(X,Y)$ $\quad B \in \mathbf{B}(Y,Z)$

then $\underset{\mathbf{B}(X,Z)}{\|B\hat{A}\|} \leq \|B\|\,\|A\|$

Suppose we have normed vector spaces X, Y

$U \subset X$ open, $f: U \to Y$ some function.

We'll say $T \in L(X,Y)$ is the derivative of f at $x \in U$, write $T = f'(x)$ if for any sequence (h_n) in X such that $\|h_n\| \to 0$, we have

$$\lim_{n\to\infty} \frac{\|f(x+h_n) - f(x) - Th_n\|}{\|h_n\|} = 0$$

(in this case, we say f is differentiable at x)

$$\left(\lim_{\|h\|\to 0} \frac{\|f(x+h) - (f(x) + Th)\|}{\|h\|} = 0 \right)$$

<u>Reality checks</u>

- if f is differentiable at $x \in U$ then it is continuous at x.
- if f is differentiable at $x \in U$ then $f'(x)$ is bounded.
- the derivative is <u>unique</u>!

<u>ex:</u> $f: \mathbb{R} \longrightarrow \mathbb{R} \quad f(x) = x^2$

derivative at x is the lin op $r \xmapsto{T} 2x \cdot r$

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - Th\|}{\|h\|} = 0 ?$$

$$\lim_{h\to 0} \frac{|(x+h)^2 - x^2 - 2x \cdot h|}{|h|} = \lim_{h\to 0} \frac{|h^2|}{|h|}$$

$$= \left| \lim_{h\to 0} |h| \right| = 0$$

$$F : C([0,1]) \longrightarrow C([0,1])$$
$$f \longmapsto f^2$$

" cont. functions $[0,1] \to \mathbb{R}$

def $\|g\| = \sup\{g(x) \mid x \in [0,1]\}$

<u>Q</u>: $F'(f) = T_{2f}$?

$$T_{2f}(g) = 2f \cdot g$$

$$\left(T_{2f}(g+h) = 2fg + 2fh = T_{2f}(g) + T_{2f}(h) \right)$$

$(h_n) \to 0$

$$\lim_{n\to\infty} \frac{\|F(f+h_n) - F(f) - T_{2f} \cdot h_n\|}{\|h_n\|} \overset{?}{=} 0$$

$$\lim_{n\to\infty} \frac{\|(f+h_n)^2 - f^2 - 2f \cdot h_n\|}{\|h_n\|}$$

$$= \lim_{n\to\infty} \frac{\|h_n^2\|}{\|h_n\|} \begin{array}{l} \leftarrow \sup\{h_n(x)^2 \mid x \in [0,1]\} \\ \leftarrow \sup\{h_n(x) \mid x \in [0,1]\} \end{array}$$

$\|h_n\| \to 0 \qquad \forall \varepsilon > 0 \ \exists N$ s.t. $n \geq N$

$\|h_n\| < \varepsilon$

$|h_n(x)| < \varepsilon$ all x

$|h_n(x)|^2 < \varepsilon^2$

$|h_n(x)^2| < \varepsilon^2$ all x