Recall: if X, Y normed vector spaces

$U \subset X$ open set, $f: U \to Y$ function, and $x \in U$ a point

we'll say that a linear transformation $T: X \to Y$ is the derivative of f at x and write $f'(x) = T$ if

$\forall\ (h_n)$ sequences in X s.t. $\lim_{n\to\infty} \|h_n\| = 0$

$$\lim_{n\to\infty} \frac{\|f(x+h_n) - f(x) - Th_n\|}{\|h_n\|} = 0$$

(Fréchet)

Remark: this is well defined! if $T, S: X \to Y$ both satisfy above then $T = S$.

Suppose that for all sequences (h_n) in X w/ $\|h_n\| \to 0$ we have

$$\lim_{n\to\infty} \frac{\|f(x+h_n) - f(x) - Th_n\|}{\|h_n\|} = 0 \quad \&$$

$$\lim_{n\to\infty} \frac{\|f(x+h_n) - f(x) - Sh_n\|}{\|h_n\|} =$$

Consider $h_n = \frac{1}{n} h$ for $h \in X$ arbitrary.

$\|h_n\| \to 0$ $\quad Th = Sh \Rightarrow T = S$

$h \neq 0$

$$\frac{\|(T-S)h\|}{\|h\|} = \frac{\|\frac{1}{n}(T-S)h\|}{\|\frac{1}{n}h\|} = \frac{\|(T-S)(\frac{1}{n}h)\|}{\|\frac{1}{n}h\|}$$

$$\underset{=}{0}$$

$$\lim_{n\to\infty} \frac{\|(T-S)h_n\|}{\|h_n\|} = \lim_{n\to\infty} \frac{\|Th_n - Sh_n\|}{\|h_n\|}$$

$$= \lim_{n\to\infty} \frac{\|f(x+h_n) - f(x) - f(x+h_n) + f(x) + Th_n - Sh_n\|}{\|h_n\|}$$

$$= \lim_{n\to\infty} \frac{\|(f(x+h_n) - f(x) - Sh_n) + (-(f(x+h_n) - f(x) - Th_n))\|}{\|h_n\|}$$

$$\leq \lim_{n\to\infty} \frac{\|f(x+h_n) - f(x) - Sh_n\| + \|f(x+h_n) - f(x) - Th_n\|}{\|h_n\|}$$

$$\leq \underbrace{\lim_{n\to\infty} \frac{\|f(x+h_n) - f(x) - Sh_n\|}{\|h_n\|}}_{0} + \underbrace{\lim_{n\to\infty} \frac{\|f(x+h_n) - f(x) - Th_n\|}{\|h_n\|}}_{0}$$

How to compute these derivatives?

Hope: Assume (at first) that $f'(x)$ exists.
compute what it would have to do. $\rightsquigarrow T$
check that T works.

Def: Given X, Y normed v. spaces, $f: U \to Y$ function, $U \subset X$ open, $x \in U$, $v \in X$, then we define the Gateaux differential of f at x along v written $df(x,v)$ to be the limit:

$$df(x,v) \equiv \lim_{r \to 0} \frac{f(x+vr) - f(x)}{r} \in Y$$

Claim: If f is differentiable at x, and $v \in X$ then

$$\boxed{f'(x)v = \frac{1}{\|v\|} df(x,v)}$$

$f: \mathbb{R}^n \longrightarrow \mathbb{R}$

$x = \vec{x} = (x_1, \ldots, x_n)$

$f(x_1, \ldots, x_n) = f(x)$

$$= \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$$

$T = f'(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$

$f'(x)v = \frac{1}{\|v\|} df(x,v)$

$[T_1 \cdots T_n]$

$1 \times n?$

$$[T_1 \cdots T_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \text{"} T \cdot x \text{"}$$

$$[T_1 \cdots T_n] \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{e_i} = T_i$$

ith spot

$$f'(x) \cdot e_i = \frac{1}{\|e_i\|} df(x, e_i) = df(x, e_i)$$

$\|$

Te_i

1

$$\lim_{r \to 0} \frac{f(x + re_i) - f(x)}{r}$$

$$= \lim_{r \to 0} \frac{f(x_1, x_2, \ldots, x_{i-1}, x_i + r, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_n)}{r}$$

i.e. fix entries $x_1, \ldots, x_{i-1}$ $x_{i+1}, \ldots, x_n$

let ith entry vary.

" $\frac{\partial f}{\partial x_i}$ " partial derivative of f w/r to ith variable.

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i.e. (standard notation)

$$f(x_1, x_2) = 3x_1^2 - x_1 x_2 + x_2^3$$

$x_1 = 0 \quad x_2 = 1$

$$f'(0,1) = \left[\left.\frac{\partial f}{\partial x_1}\right|_{0,1} \quad \left.\frac{\partial f}{\partial x}\right|_{0,1} \right]$$

$$\frac{\partial f}{\partial x_1} = 6x_1 - x_2 \qquad \frac{\partial f}{\partial x_2} = -x_1 + 3x_2^2$$

$x_1 = 0 \quad x_2 = 1$

$$\left.\frac{\partial f}{\partial x_1}\right|_{0,1} = -1 \qquad \left.\frac{\partial f}{\partial x_2}\right|_{0,1} = 3$$

$$f'(0,1) = [-1 \;\; 3]$$

more generally:

$$f(x_1, \dots, x_n) : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$f'(\underbrace{a_1, \dots, a_n}_{a}) = \left[\left.\frac{\partial f}{\partial x_1}\right|_a \quad \left.\frac{\partial f}{\partial x_2}\right|_a \quad \cdots \quad \left.\frac{\partial f}{\partial x_n}\right|_a \right]$$

$$f' = \left[\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \quad \cdots \quad \frac{\partial f}{\partial x_n} \right]$$

"the gradient"