Survey about exam date:

inconclusive   Mar 3 / 8 / 10

Last time:

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $x \in \mathbb{R}^n$, $f'(x)$ if it exists
is a lnr map $\mathbb{R}^n \to \mathbb{R}^m$

idea: $f(x+h) \approx f(x) + f'(x)(h)$

lnr trans.   vector in $\mathbb{R}^n$

formally: $(h_n) \in \mathbb{R}^n$ any sequence of vectors w/ $\|h_n\| \to 0$

$$\lim_{n \to \infty} \frac{\| f(x+h_n) - f(x) - f'(x)h_n \|}{\|h_n\|} = 0$$

$$\left[\text{imagine} \lim_{\substack{n \to \infty \\ \|h_n\| \to 0}} \frac{f(x+h_n) - f(x)}{h_n} - f'(x) \cdot \frac{\cancel{h_n}}{\cancel{h_n}} = 0 \right]$$

$$df(x,v) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(x+hv) - f(x)}{h}$$

and if $v = e_i$ "standard basis vectors"

$$df(\vec{x}, v) = \frac{\partial f}{\partial x_i}\Big|_{\vec{x}}$$

<u>Proposition</u> 8.3.9: Let $U \subset \mathbb{R}^n$ be open

$f: U \longrightarrow \mathbb{R}^m$ differentiable

$p \longmapsto (f_1(p), f_2(p) \ldots, f_m(p))$ (i.e. $f'(x)$ exists for all $p \in U$)

then for $x \in U$, $f'(p)$ is represented by the matrix

$$\begin{bmatrix} \partial f_1/\partial x_1|_p & \partial f_1/\partial x_2|_p & \cdots & \partial f_1/\partial x_n|_p \\ \vdots & & & \\ \partial f_m/\partial x_1|_p & \cdots & \cdots & \partial f_m/\partial x_n|_p \end{bmatrix}$$

eg: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$f(x_1, x_2) = (x_1 + x_2, x_1 x_2, x_2^3)$

$$
f'(x_1,x_2) = \begin{array}{c c c}
 & \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \\ \partial f_3/\partial x_1 & \partial f_3/\partial x_2 \end{bmatrix} & \\
\end{array}
$$

$$
\begin{array}{r c l}
x_2 & \partial f_2/\partial x_1 \quad \partial f_2/\partial x_2 & x_1 \\
0 & \partial f_3/\partial x_1 \quad \partial f_3/\partial x_2 & 3x_2^2
\end{array}
$$

$$
= \begin{bmatrix} 1 & 1 \\ x_2 & x_1 \\ 0 & 3x_2^2 \end{bmatrix}
$$

$$
f(1,1) = (1+1,\ 1\cdot 1,\ 1^3) = (2,1,1)
$$

$$
f'(1,1) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}
$$

$$
f(1+h_1, 1+h_2) \approx \underbrace{f(1)}_{= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}} + f'(1)\cdot \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}
$$

h_1, h_2 small

$$
\approx \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}
$$

$$
\approx \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} h_1 + h_2 \\ h_1 + h_2 \\ 3h_2 \end{bmatrix}
$$

$$f(1+h_1, 1+h_2) \approx \begin{bmatrix} 2+h_1+h_2 \\ 1+h_1+h_2 \\ 1+3h_2 \end{bmatrix}$$

Prop is related to fact:

$$f'(x)(h) = df(x,h)$$

relate to partial derivatives

<u>Familiar friends</u>

if $f,g : \underset{\mathbb{R}^n}{\overset{}{U}} \to \mathbb{R}^m$ are both differentiable, $U \subset \mathbb{R}^n$, and $\lambda \in \mathbb{R}$

then $(f+g)'(x) = f'(x) + g'(x)$ and

$(\lambda f)'(x) = \lambda f'(x)$

and if $f : U \to V$, $g : V \to \mathbb{R}^l$, $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$

f,g are differentiable, then so is $g \circ f$

and $(g \circ f)' = g'(f(x)) \cdot f'(x)$

mult of matrices or composition of lin transformations

$N : V \longrightarrow \mathbb{R}$

$N(x) = \|x\|$

$x, c \in V$

$\|x - c\|$

$\shortparallel$

want to show:

cont. at $x \in V \iff \forall \varepsilon > 0\ \exists \delta > 0$ s.t. $d(x,c) < \delta$

$\Rightarrow d(N(x), N(c)) < \varepsilon$

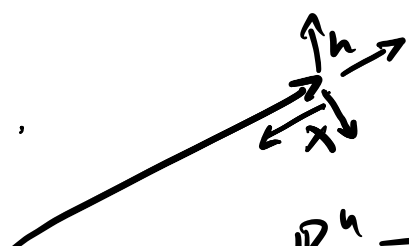
$|N(x) - N(c)|$

$|\, \|x\| - \|c\| \,|$

Sam ← HiChew

choose $\varepsilon > 0$

$N(x) = \|x\| = \sqrt{\sum x_i^2}$



$\mathbb{R}^n \to \mathbb{R}$

$$N'(x) = \frac{x^t}{\|x\|} = \frac{1}{\|x\|}[x_1, x_2\ x_3]$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad = \left[\frac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, \ \ldots \right]$$

Facts:

polynomial functions are differentiable.

i.e. if $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$f(p) = (f_1(p), f_2(p), \ldots, f_m(p))$$

where each $f_i: \mathbb{R}^n \longrightarrow \mathbb{R}$

$f_i(x_1, \ldots, x_n)$ is a poly in $x_1, \ldots, x_n$

$$x_1^5 + x_1 x_3 + x_4^7 + \ldots$$

Idea of inverse function thm:

if $f : \underset{\mathbb{R}^n}{\underset{\cap}{U}} \longrightarrow \mathbb{R}^n$ is continuously differentiable.

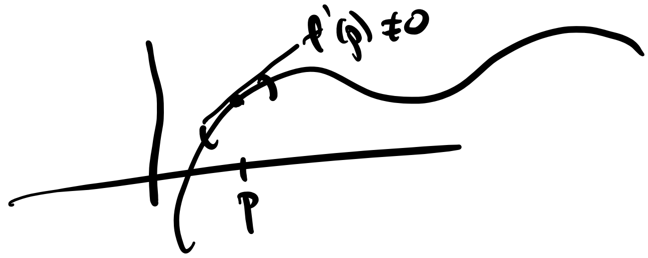
and if $f'(p)$ is invertible $(\det f'(p) \neq 0)$

then $\exists$ Ball $B = B_\varepsilon(p) \subset U$ s.t.

$$f : B \longrightarrow V = f(B) \subset \mathbb{R}^n$$

is an invertible function – i.e. $\exists\, f^{-1} : V \longrightarrow B$

also a cont. diff.



We'll do the fun part: we'll show, can find B s.t.

f is 1-1 when restricted to B.

Main ingredient: analog of mean value theorem.

Standard MVT: $\dfrac{f(b)-f(a)}{b-a} = f'(c)$ for some $c \in (a,b)$

if $f : [a,b] \to \mathbb{R}$ is cont. diff alternate (weaker) version:

if $|f'(x)| \leq M$ all $x \in (a,b)$ then

$|f(b) - f(a)| \leq M|b-a|$

general version: if $f: \mathbb{R}^n \to \mathbb{R}^n$

$\overset{\cup}{B}$ ball

operator norm

and $\|f'(x)\| \leq M$

$x \in B$

gen MVT $\Rightarrow$ $\|f(b) - f(a)\| \leq M\|b-a\|$

Given $p \in U \xrightarrow[f]{\subset \mathbb{R}^n} \mathbb{R}^n$ want to know

how many other $x \in U$ satisfy $f(x) = f(p) = y$

Construct: $\phi : \mathbb{R}^n \to \mathbb{R}^n$ $\quad A = f'(x)$

$$\phi(x) = x + A^{-1}(y - f(x))$$

$$\phi(x) = x \iff A^{-1}(y - f(x)) = 0$$

A^{-1} invertible

$\Rightarrow y - f(x) = 0$

$f(p) = y = f(x)$