Goal: Inverse function theorem

Theorem (8.5.7) Let $U \subset \mathbb{R}^n$ open $f: U \to \mathbb{R}^n$ be continuously differentiable. Suppose $p \in U$ such that $f'(p)$ (lin. trans from $\mathbb{R}^n$ to $\mathbb{R}^n$) is invertible (i.e. $\det f'(p) \neq 0$)

then $\exists\, B = B_\varepsilon(p)$ such that if we let $W = f(B)$ then $f: B \to W$ is bijective & the inverse function $f^{-1}: W \to B$ is also continuously differentiable and if $y = f(x)$ $x \in B$ then $(f^{-1})'(y) = f'(x)^{-1}$

Meaning of continuously differentiable: X, Y normed vector spaces

$$\begin{array}{ccc} U & \xrightarrow{f} & Y \\ \cap & & \\ X & & \end{array}$$

if $\forall x \in U$, $f'(x)$ exists

$$f'(x) \in L(X, Y)$$

$\rightsquigarrow f': U \longrightarrow L(X,Y)$ ← normed v. space w/ operator norm

$X = \mathbb{R}^n \qquad Y = \mathbb{R}^m \qquad f'(x) = \left[\partial f_i / \partial x_j \big|_x \right]_{i,j} \in \mathbb{R}^{nm}$

in particular, $L(X,Y)$ is a normed v.space $\Rightarrow$ it's a metric space

$$d(T_1,T_2) = \|T_1 - T_2\|$$

$$\|T\| = \sup\left\{ \|Tx\| / \|x\| \;\middle|\; \begin{matrix} x \neq 0 \\ \text{in } X \end{matrix} \right\}$$

so we can ask if $f' : X \longrightarrow L(X,Y)$ is continuous.

if it is, we say f is continuously differentiable.

(turns out the operator norm & Euclidean norm are close enough so that same topology (open sets))

$$f''(x) \in L(X, L(X,Y)) \rightarrow L(X \times X, Y)$$

$$\varphi$$

$f(x) \in Y$

$f'(x) \in L(X,Y)$

$\varphi(x, \lambda x')$

$\varphi(\lambda x, x')$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1}$$

Multivariable MVT

Theorem: if $f:[a,b] \to \mathbb{R}^n$ is differentiable
then $\|f(b)-f(a)\| \leq M\,|b-a|$ where
$M = \sup\{\|f'(x)\| \mid x \in (a,b)\}$

norm in $\mathbb{R}^n$

op. norm for $L(\mathbb{R},\mathbb{R}^n)$

norm in $\mathbb{R}$

Pf: consider the function $\varphi:[a,b] \to \mathbb{R}$

$$\varphi(t) = (f(b)-f(a)) \cdot f(t)$$

$$\|f(b)-f(a)\|^2 = (f(b)-f(a)) \cdot (f(b)-f(a))$$

$$= (f(b)-f(a)) \cdot f(b) - (f(b)-f(a)) \cdot f(a)$$

$$= \varphi(b)-\varphi(a) = (b-a)(f(b)-f(a)) \cdot f'(t)$$

$$\frac{\varphi(b)-\varphi(a)}{b-a} = \varphi'(t) \quad \text{some } t \in (a,b)$$

$$\varphi(b)-\varphi(a) = (b-a)\varphi'(t)$$

Claim $\varphi'(t) = (f(b)-f(a)) \cdot f'(t)$

$$\left(\text{in general, } f: \mathcal{U} \to \mathcal{Y} \quad T: \mathcal{Y} \to \mathcal{Z} \atop (T \circ f)' = T \circ f' \right)$$

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$v \longmapsto (f(b)-f(a)) \cdot v$$

$$\|f(b)-f(a)\|^2 = \varphi(b) - \varphi(a) = (b-a)(f(b)-f(a)) \cdot f'(t)$$

$$\left(\text{Cauchy-Schwartz: } |v \cdot w| \le \|v\| \|w\| \right.$$

Euclidean

$$|(b-a)(f(b)-f(a)) \cdot f'(t)| \le (b-a) \|f(b)-f(a)\| \, \|f'(t)\|$$

$$\|f(b)-f(a)\|^{\cancel{2}} \le (b-a) \cancel{\|f(b)-f(a)\|} \, \|f'(t)\|$$

$$\|f(b)-f(a)\| \le (b-a) \|f'(t)\|$$

$$\|f(b)-f(a)\| \le (b-a) M$$

Euclidean (operator??)

$$M = \sup\{ \|f'(t)\| \mid t \in (a,b) \}$$

One detail (HW?) consider $f: \mathbb{R} \to \mathbb{R}^n$

$$f': \mathbb{R} \to \mathbb{R}^n$$

operator norm = Euclidean norm of a $1 \times n$ matrix

$\square$

Theorem (Actual MVT)

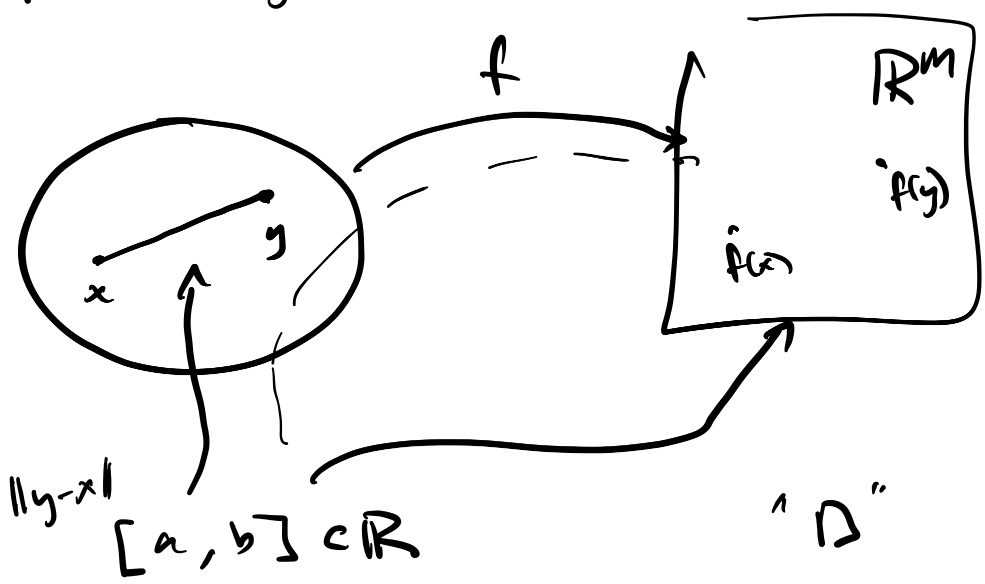
$U \subset \mathbb{R}^n \quad f: U \longrightarrow \mathbb{R}^m$ differentiable.

and suppose $U = \text{Ball} = B_r(p) \quad p \in \mathbb{R}^n$

then if $M \geq \|f'(x)\|$ all $x \in U$ then

$$\|f(x) - f(y)\| \leq M\|x-y\|$$

Pf:



$U \xrightarrow{f} \mathbb{R}^n \quad$ f cont diff

$U \subset \mathbb{R}^n$

$p \in U \quad f'(p)$ invertible.

want $B_\varepsilon(p)$ s.t. f is 1-1 restricted to $B_\varepsilon(p)$.

choose x_0 want to show no other x will satisfy $f(x) = f(x_0)$

Strategy: $\overset{x_0}{\downarrow}$ $\phi : U \longrightarrow \mathbb{R}^n$ "1-1 detector fcn"

$$\phi(x) = x \iff f(x) = f(x_0)$$

$$\phi(x) = x + f'(p)^{-1}\big(f(x_0) - f(x)\big)$$

we'll show: if $\varepsilon <<$ small then ϕ' small

if $\varphi(x_0) = x_0$ $\varphi(x) = x$

$\|\varphi(x_0) - \varphi(x)\| \leq \underset{\text{small}}{\|\varphi'\|}$ $\|x_0 - x\|$

$\|x_0 - x\|$