On the agenda for today:

7) $T: C([0,1])_1 \longrightarrow \mathbb{R}$

$T(f) = f(0)$

- Show lin. trans
- Show not bounded

14) $f(x,y) = \sin(x-y^2)$ chain rule to show f diff &, find the derivative.

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7) to show T a lin. transform. need to check

a) $T(f+g) = T(f) + T(g)$

b) $T(\lambda f) = \lambda T(f)$

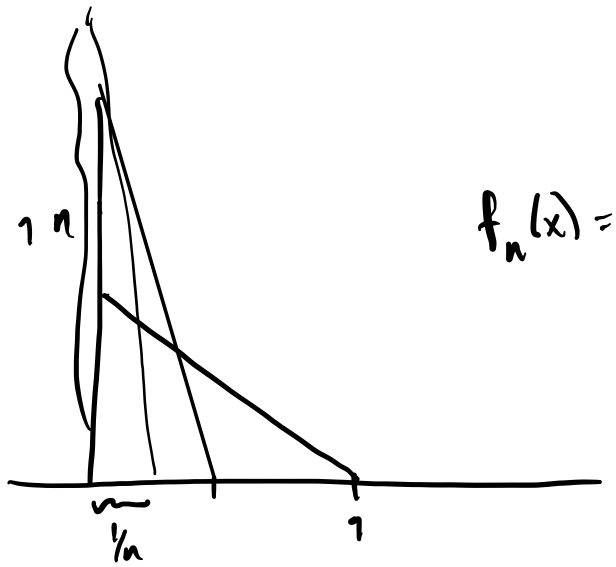
$T(f+g) = (f+g)(0) = f(0) + g(0)$
$= T(f) + T(g)$ ✓

$T(\lambda f) = (\lambda f)(0) = \lambda \cdot f(0)$
$= \lambda \cdot T(f)$ ✓

Show T not bounded.

$$\|T\| = \sup\left\{ \frac{\|T(f)\|}{\|f\|} \;\middle|\; f \in C([0,1])_1, f \neq 0 \right\}$$

$$= \sup\left\{ \frac{|f(0)|}{\int_0^1 |f(x)|dx} \;\middle|\; \text{— '' —} \right\}$$



$$f_n(x) = \begin{cases} -n^2 x + n & 0 \le x \le \frac{1}{n} \\ 0 & \text{else} \end{cases}$$

$$\int |f_n(x)|\,dx = \frac{1}{2} = \frac{1}{2} b \cdot h$$

$f(0) = n$

$$\|T\| \geq \frac{|f_n(0)|}{\int_0^1 |f_n(x)|\,dx} = \frac{n}{(1/2)} = 2n \quad \text{all } n.$$

$$\Rightarrow \|T\| = \infty.$$

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14) $f(x,y) = \sin(x - y^2)$

polys are diff.
know stuff from 1 var. calculus.

$g(x,y) = x - y^2$

$h(z) = \sin z$

$h(g(x,y)) = f(x,y)$

g = polynomial, so diff. h = diff because

$f'(x,y) = h'(g(x,y)) \cdot g'(x,y)$

$$g'(x,y) = [\,\partial g/\partial x \quad \partial g/\partial y\,] = [1 \quad -2y]$$

$h'(z) = \cos z$

$f'(x,y) = h'(g(x,y)) \cdot g'(x,y)$

$= \cos(x-y^2) \cdot [1 \quad -2y]$

Chain rule says: if g, h diff. then so is $g \circ h$

and $(g \circ h)'(p) = g'(h(p)) \cdot h'(p)$

matrix multiplication

$$\boxed{[\cos(x-y^2) \qquad -2y\cos(x-y^2)]}$$

8: e_1, e_2 standard basis vectors in $\mathbb{R}^2$

$f: \mathbb{R}^2 \to \mathbb{R}$ $\qquad f(v) = \begin{cases} \dfrac{(v\cdot e_1)^2 (v\cdot e_2)}{\|v\|^2} & v \neq 0 \\ 0 & v = 0 \end{cases}$

Compute $df(0,e_1)$ $\quad df(0,e_2)$

$df(0, e_1+e_2)$

$dg(p,v) = \lim_{h\to 0} \dfrac{g(p+hv) - g(p)}{h} = \lim_{h \to 0} \dfrac{k(h) - k(0)}{h}$

$= k'(0)$

$g: \mathbb{R}^4 \to \mathbb{R}$

$k(h) = g(p+hv)$

$df(0,e_1)$

$k(t) = f(0+e_1t) = f(e_1t) = \dfrac{(e_1t\cdot e_1)^2(\overset{\swarrow 0}{e_1t\cdot e_2})}{\|e_1t\|^2}$

$= 0$

$df(0,e_1) = k'(t) = 0$

$df(0,e_2)$

$k(t) = f(0+e_2t) = \dots = \dfrac{(\overset{\swarrow 0}{e_2t\cdot e_1})^2(e_2t\cdot e_2)}{\|e_2t\|^2}$

$= 0 \qquad k'(t) = 0 \qquad df(0,e_2)$

$df(0,e_1+e_2)$

$k(t) = f((e_1+e_2)t) = \dfrac{((e_1+e_2)t\cdot e_1)^2((e_1+e_2)t\cdot e_2)}{\|(e_1+e_2)\cdot t\|^2}$

$= \dfrac{t^2\cdot t}{t^2\|e_1+e_2\|^2} = \dfrac{t^3}{2t^2} = \dfrac{1}{2}t$

$k'(t) = \frac{1}{2} \qquad df(0,e_1+e_2) = \frac{1}{2}$

b) no, not diff.

if it was then $df(0,e_1) = f'(0)(e_1)$

e_2

$\frac{1}{2} = df(0,e_1+e_2) = f'(0)(e_1+e_2)$

$= f'(0)e_1 + f'(0)e_2$

$= df(0,e_1) + df(0,e_2) = 0+0$

If $f: \underset{\mathbb{R}^n}{\overset{\cap}{U}} \longrightarrow \mathbb{R}^m$ is differentiable at p ⟵ Frechet

then for every $v \in \mathbb{R}^n$, $df(p,v)$ also exists

Gateaux

and $df(p,v) = f'(p)(v)$

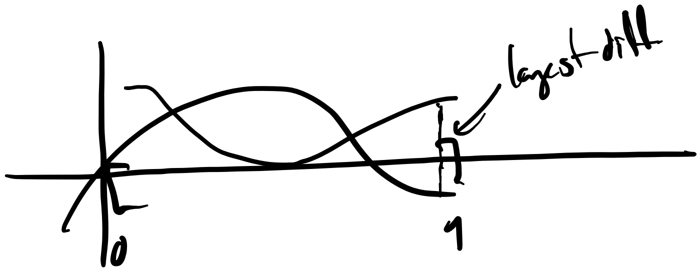
On the other hand, its possible for $df(p,v)$ to exist for all v, but $f'(p)$ to not exist.

By hand Gateaux

$$df(0,e_1) = \lim_{h\to 0} \frac{\overset{0}{f(he_1)} - \overset{0}{f(0)}}{h} = 0$$

$$df(0,e_1+e_2) = \lim_{h\to 0} \frac{\left(\frac{((e_1+e_2)h\cdot e_1)^2 \cdot ((e_1+e_2)h\cdot e_2)}{\|(e_1+e_2)h\|^2} - \overset{0}{f(0)}\right)}{h}$$

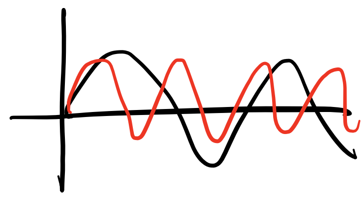
$$= \lim_{h\to 0} \frac{\left(\frac{h^2\cdot h}{2h^2}\right) - 0}{h} = \lim_{h\to 0} \frac{h^3}{2h^3} = \frac{1}{2}$$

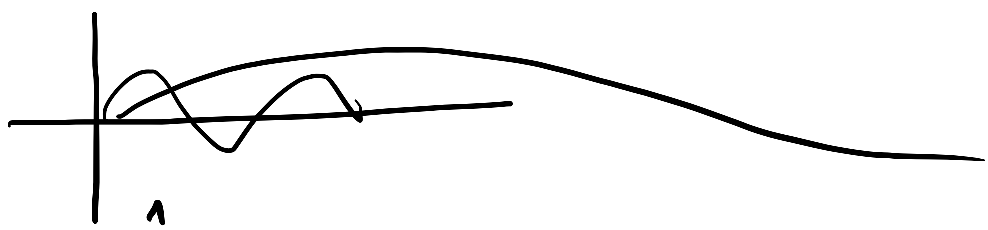


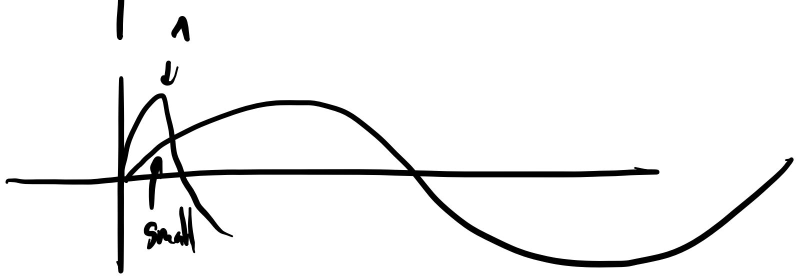
$f_n(x) = \sin(nx)$

$\|f\|_\infty = \sup\{|f(x)|\}_{x \in [0,1]}$

$d(f,g) = \|f-g\|_\infty = \sup\{|f(x)-g(x)|\}$







$\forall n \quad \exists N \quad \sin(N\cdot\varepsilon) \sim 1$

$\sin(n\varepsilon) \sim 0$

$f_n \quad \sin(nx) \quad x = \frac{\pi}{2n} \quad \sin(nx) = \sin\left(\frac{\pi}{2}\right) = 1$

$f_{2n} \quad \sin(2nx) \quad [0,1] \quad n \gg 0 \quad \sin(2nx) = \sin(\pi) = 0$

$$\|f_n - f_{2n}\| = d(f_n, f_{2n}) \geq 1$$

$$\geq \left| f_n\left(\tfrac{\pi}{2n}\right) - f_{2n}\left(\tfrac{\pi}{2n}\right)\right| = 1$$

$$f(x,y) = (e^x \sin y,\ x - 2y)$$

if it is diff, then the derivative would be

$$\begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix} = \begin{bmatrix} e^x \sin y & e^x \cos y \\ 1 & -2 \end{bmatrix}$$