Reminder:

(7.1)

Def A metric on a set X is a function $d: X \times X \to \mathbb{R}$ such that

1) $d(x,y) \geq 0$
2) $d(x,y) = 0 \iff x = y$
3) $d(x,y) = d(y,x)$
4) $d(x,y) + d(y,z) \geq d(x,z)$

Warm-up problems

X = set of strings of 3 digits

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$$d_1(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x \,\&\, y \text{ share first 2 digits} \\ 2 & \text{if } x \,\&\, y \text{ share first digit only} \\ 3 & \text{else} \end{cases}$$

order does matter

$$d_2(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if share all digits in common, not same} \\ 2 & \text{if share any 2 digits} \\ 30 & \text{if 1 digit} \\ 40 & \text{else} \end{cases}$$

order doesn't matter

$$\underset{x}{123} \overset{2}{\frown} \underset{y}{234} \overset{30}{\frown} \underset{z}{456}$$

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First goal:

Take a closer look at the Euclidean distance function

OH: 5:10 - 6pm Thurs

$X = \mathbb{R}^n$ $\qquad d(\vec{x},\vec{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

$$= \|\vec{x} - \vec{y}\|$$

$$= \sqrt{(\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y})}$$

$\vec{v} = (v_1, \ldots, v_n)$ $\qquad \vec{v} \cdot \vec{v} = \sum v_i^2 = \|v\|^2$

Check that this is a metric

$\mathbb{R}^2$ or $\mathbb{R}^3$

$$d(u,w) \leq d(u,v) + d(v,w)$$

u, v, w $\qquad \|u-w\| \overset{?}{\leq} \|u-v\| + \|v-w\|$

$$\|u-w\|^2 \underset{?}{\leq} \|u-v\|^2 + 2\|u-v\|\,\|v-w\| + \|v-w\|^2$$

$$(u-w)\cdot(u-w)$$

$$(u-v)\cdot(u-v) + 2\|u-v\|\|v-w\| + (v-w)\cdot(v-w)$$

$$u-w = (u-v)+(v-w)$$

$$\big((u-v)+(v-w)\big)\cdot\big((u-v)+(v-w)\big)$$

$$= (u-v)\cdot(u-v) + 2(u-v)\cdot(v-w) + (v-w)\cdot(v-w)$$

comes down to showing

$$2\underbrace{(u-v)}_{\vec{x}}\cdot\underbrace{(v-w)}_{\vec{y}} \leq 2\|\underbrace{u-v}_{\vec{x}}\|\|\underbrace{v-w}_{\vec{y}}\|$$

$$\vec{x}\cdot\vec{y} = \|\vec{x}\|\|\vec{y}\|\cos\theta \quad \Uparrow \quad |\cos\theta| \leq 1$$

Challenge: $|\vec{x}\cdot\vec{y}| \leq \|\vec{x}\|\|\vec{y}\|$ w/out trig.

$$\vec{x} = (x_1, x_2)$$

$$|x_1y_1 + x_2y_2| \quad \sqrt{(x_1^2+x_2^2)(y_1^2+y_2^2)}$$

$$\sqrt{\cancel{x_1^2y_1^2} + \cancel{x_2^2y_2^2} + x_1^2y_2^2 + x_2^2y_1^2}$$

$$\geq$$

$$(x_1y_1+x_2y_2)^2 \overset{?}{\leq}$$

$$\cancel{x_1^2y_1^2}+2x_1y_1x_2y_2+x_2^2\cancel{y_2^2} \leq (\qquad\qquad)$$

$$0 \leq x_1^2y_2^2 - 2x_1y_1x_2y_2 + x_2^2y_1^2$$

$$0 \leq (x_1y_2 - x_2y_1)^2$$

WTS: $|\vec{x}\cdot\vec{y}| \leq \|\vec{x}\|\,\|\vec{y}\|$ "Cauchy-Schwartz Inequality"

$$(\vec{x}\cdot\vec{y})^2 \leq \|\vec{x}\|^2\|\vec{y}\|^2$$

$$\left(\sum x_iy_i\right)^2 \leq \left(\sum x_i^2\right)\left(\sum y_j^2\right)$$

$$2\left(\sum x_iy_i\right)^2 \leq 2\left(\sum x_i^2\right)\left(\sum y_j^2\right)$$

$$2\left(\sum x_i^2\right)\left(\sum y_i^2\right) - 2\left(\sum x_iy_i\right)^2 \overset{?}{\geq} 0$$

$$\left(\sum x_i^2\right)\left(\sum y_j^2\right) + \left(\sum x_j^2\right)\left(\sum y_i^2\right)$$

$$-2\left(\sum x_iy_i\right)\left(\sum x_jy_j\right)$$

$$\sum_{i,j} x_i^2y_j^2 + x_j^2y_i^2 - 2x_iy_jx_jy_i$$

$$= \sum_{i,j} (x_i y_j - x_j y_i)^2 \geq 0 \quad \square.$$

Definition A metric space is a pair (X,d) consisting of a set X and a metric d on X.

If (X,d) is a metric space, and $Y \subset X$ any subset

then can consider restriction of d to $Y \times Y$

this gives a metric on Y. $(Y, d|_Y)$

we say that $(Y, d|_Y)$ is a subspace of (X,d)

Abuse of notation: often write X for (X,d)

$d: X \times X \longrightarrow \mathbb{R}$ if $Y \subset X$

$Y \times Y \subset X \times X$

$d|_Y : Y \times Y \to \mathbb{R}$

defined by $d|_Y(y_1, y_2) = d(y_1, y_2)$

ex: $\mathbb{R}^2 \subset \mathbb{R}^3$

$\mathbb{R}^2 = \{(x,y,0)\} \subseteq \mathbb{R}^3$

$\mathbb{R}^2$ w/ standard Euclidean metric is a subspace of $\mathbb{R}^3$ w/ standard Eucl. metric.

Sequences & Convergence

Def if X is a metric space then a sequence in X is a function

$$\mathbb{Z}_{>0} \longrightarrow X \qquad \text{"}\mathbb{N}\text{"}$$
$$n \longmapsto a_n$$

denoted $(a_n)_{n \in \mathbb{Z}_{\geq 1}}$

Def if (a_n) is a sequence in X we say that it converges to $a \in X$ and write $\lim_{n\to\infty} a_n = a$

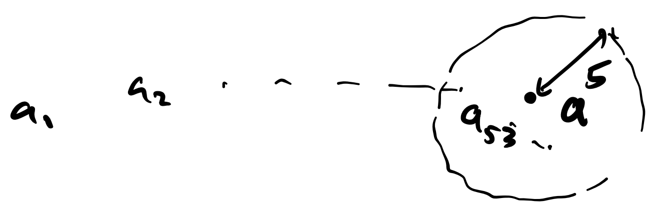
if $\forall \varepsilon > 0\ \exists N_{>0}$ integer s.t. $\forall n \geq N \Rightarrow d(a_n, a) < \varepsilon$.

Def A subset $S \subset X$ is bounded if $\exists a \in X$ and $R \in \mathbb{R}$ s.t. $d(s,a) \leq R$ for all $s \in S$.

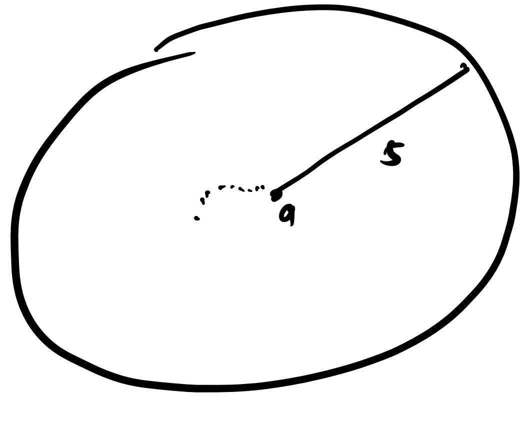
Def★ A subset $S \subset X$ is bounded★ if $\forall a \in X$ $\exists R \in \mathbb{R}$ s.t. $d(s,a) \leq R$ all $s \in S$

these are the same! (exercise)

Prop If (a_n) converges to $a \in X$ then it is bounded. i.e. the set $\{a_n \mid n \in \mathbb{Z}_{>0}\}$ is bounded in X.



$$R = \max\left(\{d(a,a_i) \mid i=1,\dots,52\} \cup \{5\}\right)$$



Pf: Suppose $\lim_{n \to \infty} a_n = a$ choose $\varepsilon = 5$

By def of lim, $\exists N > 0$ s.t. $\forall n \geq N$, $d(a_n, a) < 5$

Set $R = \max\left(\{ d(a_i, a) \mid i = 1, \dots, N-1 \} \cup \{5\} \right)$

we have $\forall i$ $d(a_i, a) \leq R$ $\square$.

Prop If (a_n) is a sequence in X and which converges to a & converges to b then $a = b$.

Pf strategy: to show $a = b$ we'll show $\forall \varepsilon > 0$ $d(a,b) \leq \varepsilon$ which implies $d(a,b) = 0$

given such an ε, since $(a_n) \to a$ $\exists N_1$ s.t. $n > N_1 \Rightarrow d(a_n, a) \leq \frac{\varepsilon}{2}$

similarly $\exists N_2$ s.t. $n > N_2 \Rightarrow d(a_n, b) \leq \frac{\varepsilon}{2}$

Let $N = \max\{N_1, N_2\}$ then for $n = N+1$

$$d(a,b) \leq d(a, a_n) + d(a_n, b) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\square$.