$\mathbb{Z}_{\geq 0}$ = all possible states of game when it's A's turn

$A \subset \mathbb{Z}_{\geq 0}$ all states in which player A wins

$B \subset$ - - - B wins

$1 \in A$ $2 \in A$ $3 \in B$ because

$\boxed{B = 3\mathbb{Z}}$

Claim

If n pennies & A's turn, who wins?

A: $n \in A$ B: $n \in B$

Goal from last time:

Prop 11.2.7 X, Y metric spaces, Y complete

(f_n) sequence in $Fun(X, Y)$ s.t. $f_n \to f$ converge uniformly

let (x_k) sequence in X s.t. $x_n \to x$

suppose $\lim_{k\to\infty} f_n(x_k)$ exist for all n. (not necessary $= f_n(x)$)

Then $\lim_{k\to\infty} \lim_{n\to\infty} f_n(x_k) \left(= \lim_{k\to\infty} f(x_k)\right)$

$= \lim_{n\to\infty} \lim_{k\to\infty} f_n(x_k)$

in particular, these limits exist

Prop X set Y complete metric space w/ (f_n) unif. Cauchy in $\mathrm{Fun}(X,Y)$ then $f_n \to f$ uniformly for some $f \in \mathrm{Fun}(X,Y)$

Using this, let's prove prop above

Pf: first want to show that $\lim\limits_{n\to\infty} \lim\limits_{k\to\infty} f_n(x_k)$ exists.

recall $\lim\limits_{k\to\infty} f_n(x_k) = a_n$ exists all n. want $\lim\limits_{n\to\infty} a_n$ exists.

Since Y complete, we just need (a_n) is a Cauchy sequence

$$d(a_n,a_m) \le d(a_n, f_n(x_k)) + d(f_n(x_k), f_m(x_k)) + d(f_m(x_k), a_m)$$

for any k

Since $f_n \to f$ uniformly, given $\varepsilon > 0$ $\exists M$ s.t. $n,m \geq M$ $d(f_n(p), f_m(p)) < \varepsilon$ all $p \in X$

in particular, is true for $p = x_k$ any x_k

$$d(a_n,a_m) \le d(a_n, f_n(x_k)) + d(f_m(x_k), a_m) + \varepsilon$$

all k.

$$\Rightarrow d(a_n,a_m) \le \underbrace{\lim_{k\to\infty} d(a_n, f_n(x_k)) + d(f_m(x_k), a_m)}_{0} + \varepsilon$$

$d(a_n, a_m) \leq \varepsilon.$

$\Rightarrow a_n$ are cauchy $\Rightarrow \underset{\substack{n\to\infty \\ \| \\ a}}{\lim} a_n = \lim_{n\to\infty} \lim_{k\to\infty} f_n(x_k)$ exists

need to show

$\lim_{k\to\infty} \underbrace{\lim_{n\to\infty} f_n(x_k)}_{f(x_k)} = a$

$f_n \to f$ uniformly

$\lim_{k\to\infty} f(x_k) = a.$

$d(f(x_m), a) \leq d(f(x_m), f_k(x_m)) + d(f_k(x_m), a_k) + d(a_k, a) < \varepsilon$

$m \gg 0$ for any k

given $\varepsilon > 0$ choose k large so that $d(f(p), f_k(p)) < \frac{\varepsilon}{3}$ all p

and also large enough so that $d(a_k, a) < \frac{\varepsilon}{3}$

and because $\lim_{m\to\infty} f_k(x_m) = a_k$

$\exists N$ s.t. $m \geq N$

$d(f_k(x_m), a_k) < \varepsilon/3$ $\square$

"Reminder" from last semester

f_n sequence of Riemann integrable functions $f_n:[a,b]\to\mathbb{R}$ converging uniformly to $f:[a,b]\to\mathbb{R}$ then

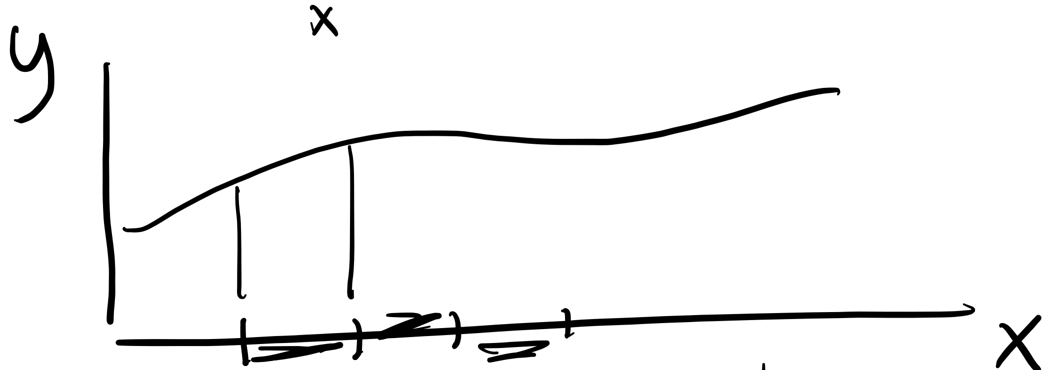
f is Riemann Integrable and

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx$$

cont...

Q: If $f: X\to Y$ function X, Y metric spaces

what is $\int_X f = ?$



✓ break X into chunks

see how big/small values are — need more than metric space

lower sum = total area in rectangles w/ small

upper - - - - - ______ large

"need 'areas of bases'" × ht of rectangle

limit over subdivisions of X

hope the answers converge to same thing.

__extra structure:__ generally want Y to be a complete normed vector space

X "measure space"

metric space w/ measure

normed v. space

__Meta theorem__ generally $f: X \to Y$ continuous is integrable if $f: X \to Y \xrightarrow{\|\;\|} \mathbb{R}$

this is integrable.

$$\begin{tikzcd}
{[a,b]} \arrow[r] \arrow[rr, bend right, "f"'] & \mathbb{R}^n \arrow[r, "f"] & \mathbb{R}
\end{tikzcd}$$

$f: X \to Y$

$\|f\|: X \to \mathbb{R}$

$\|f\|(x) = \|f(x)\|$

Y normed v. space

$Y \to \mathbb{R}$