# Lecture 22

"Real analysis" $\longleftrightarrow$ properties of the real numbers

sequences & convergence

$(a_n) : \mathbb{N} \to \mathbb{R}$

natural to also consider real functions $f : [a,b] \to \mathbb{R}$

$\mathbb{R} \to \mathbb{R}$

continuity / convergence    rates of change / derivatives    measure / integral

$\mathbb{R} \rightsquigarrow$ real functions $\rightsquigarrow$ ?

$\mathrm{Fun}(\mathbb{R},\mathbb{R})$

How did we study $\mathbb{R}$ in the first place?

continuity / convergence $\leftrightsquigarrow$ metric space.

is $\mathrm{Fun}(\mathbb{R},\mathbb{R})$ a metric space?

~~totally unjustified~~

not exactly in a useful / obvious way,

but very close in many ways

$C([0,1],\mathbb{R})$ is a metric space in many ways!

$$\|f\|_\infty = \sup\{|f(x)| \mid x \in [0,1]\}$$

$$\|f\|_p = \sqrt[p]{\int_0^1 |f(x)|^p \, dx}$$

$p \geq 1$

Differentiation?

$f: \mathbb{R} \to \mathbb{R}$ $\quad$ $f: X \to Y$

if $X$ & $Y$ are normed vector spaces, can talk about differentiability, derivatives, partial derivatives

Frechét $\quad$ Gateaux

$\mathrm{Fun}(\mathbb{R}, \mathbb{R})$ $\quad$ $C([0,1], \mathbb{R})$

Integration: $\quad$ $f: X \to Y$ $\quad$ $\int_X f \, dx$

$f: [0,1] \to \mathbb{R}$

$\int_0^1 f(x)dx \quad \int_{x \in [0,1]} f(x)dx$

write $X = X_1 \cup X_2 \cup \ldots \cup X_N$ $\quad$ $a_i \in X_i$ $\quad$ $f(a_i)$

$$\lim_{N \to \infty} \sum_{i=1}^{N} f(a_i) \cdot \underbrace{(\text{size of } X_i)}_{?}$$

? notion of a "measure"

$X$ is a "measure space"

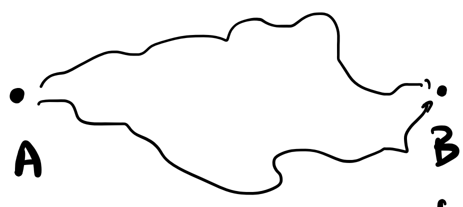
$Y$ complete normed vector space.

$f : \mathrm{Fun}(\mathbb{R},\mathbb{R}) \longrightarrow \mathbb{R}$

on some nice

functions.

Quantum mechanics



A B

t=1

t=0

"path integral formulation of QM"

$$\mathrm{prob}(A \text{ to } B) \sim \int_{\substack{\text{paths } p \\ \text{from } A \text{ to } B}} \underbrace{e^{i/\hbar\, S(p)}}_{I(p)}\, dp$$

$$I : \text{Path} \longrightarrow \mathbb{R}$$

$$\text{Path} = \mathrm{Fun}([0,1], \text{Space})$$

Options pricing.

$$\frac{dP_t}{dt} = f(P_t) + \text{noise}$$

Brownian motion

$$P_t = \int_{s=0}^{t} f(P_s)\, ds + \int_0^t \text{noise}\, dt$$

can think of this

as some integral over all

possible noise w/ weights

$$\underbrace{\int \text{noise}}_{\text{Itô integral}} = \int_{\substack{\text{all possible} \\ \text{small} \\ \text{noisy} \\ \text{trajectories}}}$$

Nobel!

Scholes - Merton

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$$\mathrm{Fun}(\mathbb{R}^3, \mathrm{Fun}(\underbrace{\mathbb{R}^2}_{\mathbb{R}^2}, \mathbb{R})) \overset{\text{bijection}}{=} \mathrm{Fun}(\mathbb{R}^5, \mathbb{R})$$

$$\cap$$

$$\mathbb{R}^5 \times \mathbb{R}$$

$$\mathrm{Fun}(X \times Y, Z) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Fun}(X, \mathrm{Fun}(Y, Z))$$

$(X \times Y) \times Z$ $\qquad$ $X \times (Y \times Z)$

$\alpha(f) \equiv \alpha(f)(x) : Y \to Z$

$\alpha(f)(x)(y) = f(x,y)$

$\beta(g) \equiv \beta(g)(x,y) = g(x)(y)$

$\alpha(\beta(g)) = g \qquad \beta(\alpha(f)) = f.$

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Arzela - Ascoli theorem.

$\mathbb{R} \to \mathrm{Fun}(\mathbb{R}, \mathbb{R})$

$\mathbb{R}^2 \to \mathbb{R}$

sequences of fcns

$\mathbb{N} \to \mathrm{Fun}(\mathbb{R}, \mathbb{R})$

$\mathbb{N} \times \mathbb{R} \to \mathbb{R}$

$f_n(x)$

<u>Thm</u>

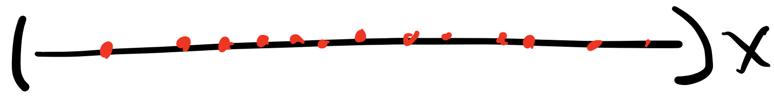
Given sequence of functions

$f_n : X \to \mathbb{R}$ $\quad$ $X$ compact metric space

such that $(f_n(x))_n$ is bounded for each $x$

"pointwise bounded"

then $\exists$ a unif. convergent subsequence if $f_n$'s are uniformly equicontinuous.

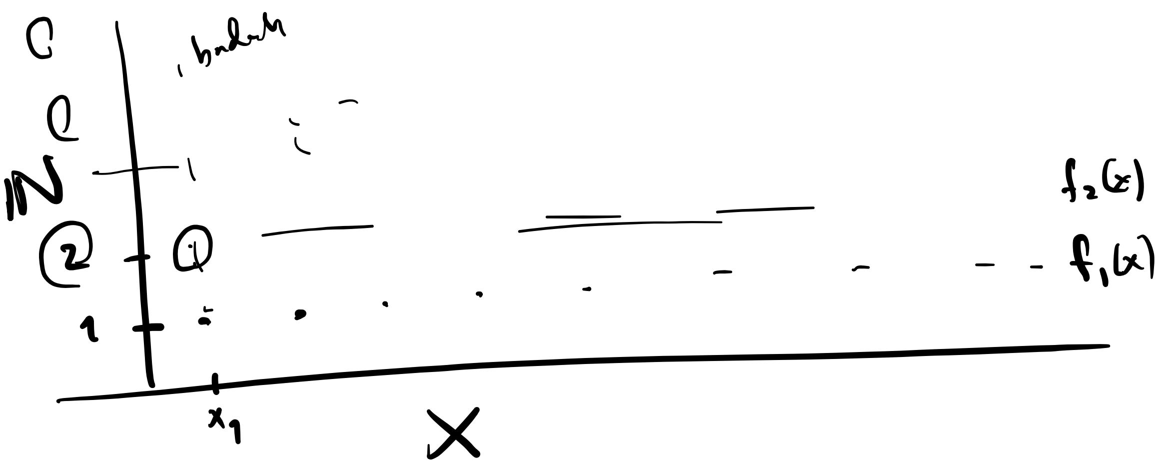



morally: replace $X$ by a dense countable subset

Core of argument: case that $X$ is countable.

i.e. $X = \mathbb{N}$

**Prop** if $X$ is a countable set, $f_n : X \to \mathbb{R}$ is a sequence of ptwise bounded fcns then $f_n$ has a subseq which pointwise converges.




$f_2$