Cleanup: official def. of convergence: we say (x_n) approaches x in a metric space X if $\forall \varepsilon>0\ \exists N>0$ s.t. $\forall n\geq N$, $d(x_n,x)<\varepsilon$.

Today: Section 7.3 (Introduction to topology)

Given (X,d) $\rightsquigarrow$ new metric by (X,d')

$$d'(x,y)=\tfrac{1}{2}d(x,y)$$

$$\rightsquigarrow\ d''(x,y)=\begin{cases} d(x,y) & \text{if } d(x,y)\leq 1\\ 1 & \text{else.}\end{cases}$$

these don't change "small scale" properties of what we do with a metric

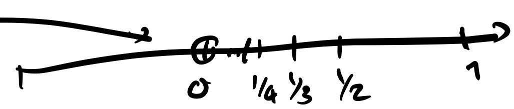
but some things are affected: in d'' every set is bounded

convergence is same with these.

We want a framework to make sense of convergence

roughly speaking, we want to understand the notion of the "edge" or "boundary" of a set

$S, S'\subseteq \mathbb{R}$ $\quad S=\{1/n \mid n\in\mathbb{Z}_{>0}\}$



0 on boundary of S & S' $\longrightarrow$ $S'=(0,1]$

Def: If X is metric space, $a \in X$, $r > 0$

$B_r(a) = \{ b \in X \mid d(a,b) < r \}$

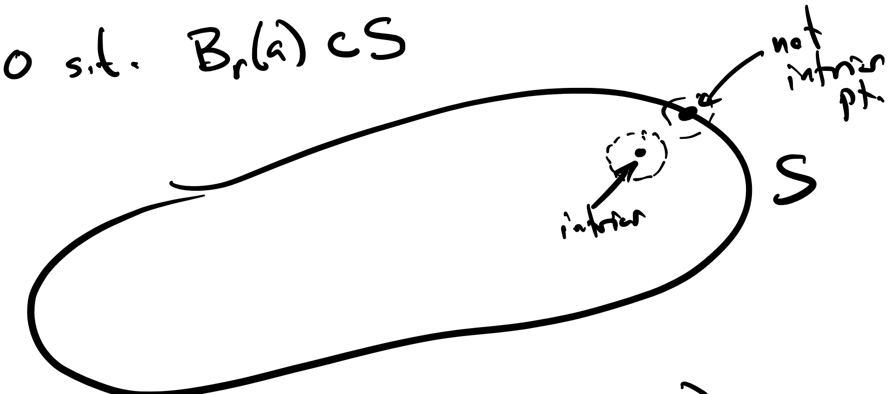
"the ball of radius r around a"

ex $X = \mathbb{R}$

$B_r(a) = (a-r, a+r)$

Def If X is a metric space, $S \subset X$ is a subset, we say $a \in S$ is in the interior of S if

$\exists\, r > 0$ s.t. $B_r(a) \subset S$

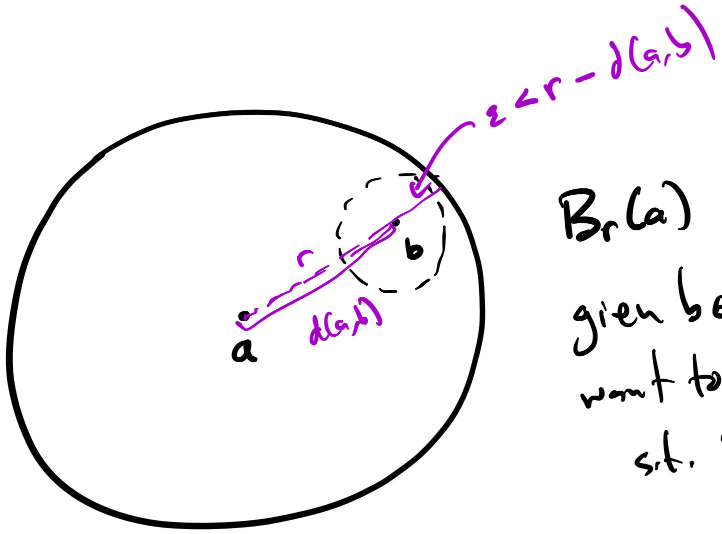


Def $S^\circ = \{ s \in S \mid s \text{ in the interior of } S \}$

$\partial S = S \setminus S^\circ$ "boundary of S"

Def We define a subset $S \subset X$ to be __open__ if all its points are interior pts – i.e. $S = S^\circ$

ex: $(a,b) \subset \mathbb{R}$ is open $[a,b)$ not open



$B_r(a)$

given $b \in B_r(a)$
want to show $\exists \varepsilon > 0$
s.t. $B_\varepsilon(b) \subset B_r(a)$

$B_r(a)$ = "open ball of radius r about a"

$C_r(a)$ = "closed - - - - - "

$= \{ b \in X \mid d(a,b) \leq r \}$

<u>Def</u> A subset $S \subset X$ is closed if $X \setminus S$ is open.

<u>Proposition</u> (~7.3.3) Let X be a metric space. Then $S \subset X$ is closed if and only if whenever a sequence (s_n) $s_n \in S$ converges to a point $x \in X$, we have $x \in S$.

<u>Moral</u>: S is closed if things in S can't approach things outside of S.

Pf: Suppose (s_n) a sequence in S which convgs to $x \in X \setminus S$. We want to show that S cannot be a closed set.

Suppose it is closed, i.e. that $X \setminus S$ is open.

then $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset (X \setminus S)$

But this means (s_n) can't actually converge to x!

because $\forall n$, $s_n \in S$ so $s_n \notin X \setminus S$

$\Rightarrow s_n \notin B_\varepsilon(x)$

$\Rightarrow d(x, s_n) \geq \varepsilon$!

Conversely, suppose that for all sequences (s_n) in S s.t. $\lim s_n$ exists in X, we have $\lim s_n \in S$,

WTS, S is closed. (i.e. $X \setminus S$ open).

Choose $x \in X \setminus S$. want $\exists$ ball of some radius about x, entirely in $X \setminus S$.

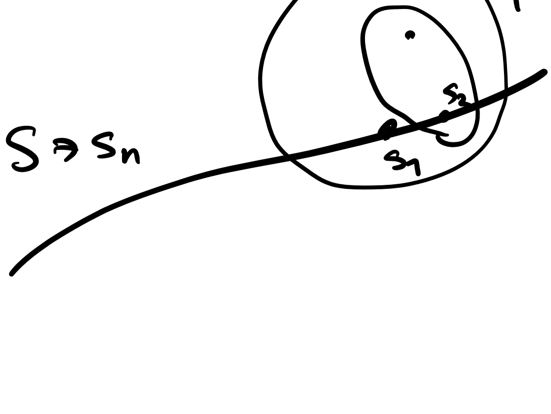
Suppose no such ball exists.

for each $n > 0$

consider $B_{1/n}(x)$

by assumption $B_{1/n}(x) \cap S \ni s_n$

$\exists\, s_n \in S$,



but now $\lim s_n = x$ $\quad \forall \varepsilon > 0, \exists N$ s.t. $n \geq N$

choose $N > \frac{1}{\varepsilon}$

$d(s_n, x) < \varepsilon$ ✓

$s_n \in B_{1/n}(x) \subset B_{1/N}(x) \quad n \geq N \quad \frac{1}{N} \leq \frac{1}{n}$

$B_{1/N}(x) \subset B_\varepsilon(x) \quad N > \frac{1}{\varepsilon} \quad \frac{1}{N} < \varepsilon$

$\square$

We say: "A set is closed if it contains all its limit points"

$X \supset \emptyset$ (open) $\qquad X \setminus \emptyset = X$ (closed)

X = open

ex: $\{1/n \mid n \in \mathbb{Z}_{>0}\} \subset \mathbb{R} = X$

$= Y$

Y is also a metric space by restricting the metric.

$\mathbb{Z}_{>0} \subset \mathbb{R}$

↑ also a metric space.

$f : \mathbb{R} \longrightarrow \mathbb{R}$

$\{f : \mathbb{R} \to \mathbb{R}\} \longrightarrow \mathbb{R}$

model of same system

performance.

$\{f : \mathbb{R} \to \mathbb{R}\} \longrightarrow \{f : \mathbb{R} \longrightarrow \mathbb{R}\}$