Def A topology on a set X is a specification of a collection of subsets of X, called "open sets" such that:

1. X, ϕ are open
2. Arbitrary unions of open sets are open
3. Finite intersections of open sets are open.

Def A topological space is a set X together with a topology on it.

As we've seen, given (X,d) a metric space, we can then define a topology on X by declaring open set = open sets

i.e. $S \subset X$ s.t. $S^0 = S$

Def if X is a top. space, (x_n) a sequence

we say $\lim\limits_{n\to\infty}^{top} x_n = x$ if $\forall\, U$ open containing x, $\exists\, N > 0$

s.t. whenever $n \geq N$, $x_n \in U$

$(U \longleftrightarrow B_\varepsilon(x))$

(Side remark if X is a general top space, limits need not be unique)

Prop: If X is a metric space (a_n) is a sequence, $a \in X$

then $\lim\limits_{n\to\infty} a_n = a \iff \lim\limits_{n\to\infty}^{top} a_n = a$

Pf: Suppose $\lim^{top}_{n\to\infty} a_n = a$ want to show $\lim_{n\to\infty} a_n = a$

Choose $\varepsilon > 0$, WTS $\exists\, N > 0$ s.t. $n \geq N \Rightarrow d(a_n, a) < \varepsilon$

By def $\lim^{top}$ $\forall\, U \ni a$ open $\exists\, N > 0$ s.t. $n \geq N \Rightarrow a_n \in U$

let $U = B_\varepsilon(a)$ open $a \in U = B_\varepsilon(a)$

so $\exists\, N > 0$ s.t. $n \geq N \Rightarrow a_n \in U = B_\varepsilon(a)$.

$d(a, a_n) < \varepsilon$ $\square \frac{1}{2}$

Suppose $\lim_{n\to\infty} a_n = a$ WTS $\lim^{top}_{n\to\infty} a_n = a$

choose $U \ni a$ open. $\Rightarrow U = U^\circ \Rightarrow a \in U^\circ$

$\Rightarrow \exists\, \varepsilon > 0$ s.t. $B_\varepsilon(a) \subset U$.

by def of lim, $\exists\, N$ s.t. $n \geq N$, $d(a, a_n) < \varepsilon$

$\Rightarrow a_n \in B_\varepsilon(a) \subset U$

therefore given U showed $\exists N$ s.t. $n \geq N$

$d(a, a_n) < \varepsilon \Rightarrow \dots \Rightarrow a_n \in U$ (def of $\lim^{top}$)

$\square \frac{2}{2}$.

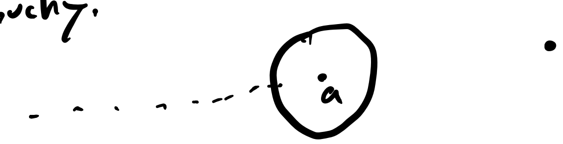
Main theme today: Convergence, compactness.

Aside: Completeness

Def If X is a metric space we say a sequence (a_n) is Cauchy if $\forall \varepsilon > 0$ $\exists N$ s.t. $\forall i,j \geq N$ then $d(a_i, a_j) < \varepsilon$.

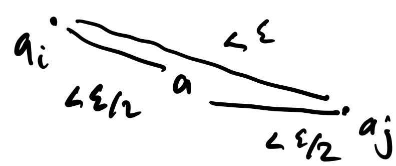
Prop: If a sequence (a_n) in a metric space X converges in X then it is Cauchy.

Pf (Illustration)



given $\varepsilon > 0$ choose N s.t. for $n \geq N$ $d(a, a_i) < \frac{\varepsilon}{2}$

$\Rightarrow$ if $i,j \geq N$



$\square$

Def X is complete if every Cauchy sequence converges!

$\mathbb{R}$ complete
$\cup$
$\mathbb{Q}$ not complete.

$a_1, \ldots, a_n, \ldots \to \sqrt{2}$

$\mathbb{Q}$ $\searrow q$

$\forall \varepsilon > 0$ $\exists N$ st $n \geq N$ $|a_n - q| < \varepsilon$

$\Rightarrow$ regarded in $\mathbb{R}$ $a_n \to q$

Prop: $\mathbb{R}^n$ w/ Euclidean metric is complete.

Proof: suppose $(\vec{a}_i)$ is a Cauchy sequence.

$$\vec{a}_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,n})$$

Claim: each sequence $a_{1,j}, a_{2,j}, \ldots$ is Cauchy.

Pf: choose $\varepsilon > 0$

$\exists N > 0$ s.t. $\forall k, l \geq N$ $d(\vec{a}_k, \vec{a}_l) < \varepsilon$

i.e. $\sqrt{\sum_{j=1}^{n} (a_{k,j} - a_{l,j})^2} < \varepsilon$

$$\sqrt{\sum (a_{k,j} - a_{l,j})^2} \geq \sqrt{(a_{k,j} - a_{l,j})^2} = |a_{k,j} - a_{l,j}|$$

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$\exists N > 0$ s.t. $\forall k, l \geq N$, $|a_{k,j} - a_{l,j}| < \varepsilon$

$\Rightarrow$ component sequences are Cauchy!

$\Rightarrow$ they each converge (because $\mathbb{R}$ is complete)

$$\lim_{i \to \infty} a_{i,j} = b_j \in \mathbb{R}$$

now, claim $\vec{a}_i \longrightarrow \vec{b} = (b_1, \ldots, b_n)$

Choose $\varepsilon > 0$ want to show $\exists N > 0$

s.t. $i \geq N$, $d(\vec{b}, \vec{a}_i) < \varepsilon$

since components conve, $a_{ij} \longrightarrow b_j$

for each $j = 1, \dots, n$ $\exists N_j$ s.t. $i \geq N_j$

$d(a_{ij}, b_j) < \frac{\varepsilon}{n}$

$\quad\; \parallel$

$|a_{ij} - b_j|$

set $N = \max \{N_j\}$

for $i > N$ $\quad d(\vec{a}_i, \vec{b}) = \sqrt{\sum_{j=1}^{n} (a_{ij} - b_j)^2}$

$$= \sqrt{\sum \frac{\varepsilon^2}{n^2}} = \sqrt{\frac{\varepsilon^2}{n}} = \frac{\varepsilon}{\sqrt{n}} < \varepsilon$$

$\square$.

If X a metric space, $S \subset X$ subset, $U_i \subset X$ are other subsets

we say U_i cover S if $S \subset \bigcup U_i$

we say U_i are an open cover of S if they're each open

Compactness

Def A subset $S \subset X$ of metric space is compact if it satisfies either of the following equivalent conditions:

1. every open covering $\{U_i\}$ of S admits a finite subcovering. i.e. $\exists$ finite set $U_{i_1}, U_{i_2}, \ldots, U_{i_r}$ which covers S.

2. every covering of S by (open) balls admits a finite subcovering.

Def If X is a metric space, a subset $S \subset X$ is called sequentially compact if any sequence (a_n) in S has a subsequence $a_{n_1}, a_{n_2}, \ldots$ which converges in S.

$n_1 < n_2 < \ldots$

Theorem $S \subset X$ is compact if and only if it is sequentially compact.

Theorem $S \subset \mathbb{R}^n$ is compact if and only if it is closed & bounded.

First targets:

Prop: if $S \subset X$ is compact then S is closed & bounded.

Partial proof Illustration:

suppose S is compact. why is it bounded?

Choose any $a \in X$ WTS $\exists\, r > 0$ s.t.

$S \subset B_r(a)$.

Consider sequence of balls

$B_1(a) \subset B_2(a) \subset B_3(a) \ldots$

$S \subset \bigcup_{n \geq 1} B_n(a) = X$