

Roughly: a stack  $\sim$  sheaf of categories

ex:  $X$  top space  $\quad \text{Shv}_X : \text{Open}(X)^{\text{op}} \rightarrow \text{Cat}$   
 $U \longmapsto \text{Shv}(U)$

given  $U \xrightarrow{i} V$ ,  $\mathcal{F} \in \text{Shv}(V)$ , can consider  $i^* \mathcal{F} \in \text{Shv}(U)$

$$\text{Shv}_X(i) : \text{Shv}(V) \xrightarrow{i^*} \text{Shv}(U)$$

Q: is this a presheaf of Cat's?

$$\text{Cat} = \text{Cat. of } \hat{\mathcal{A}} \text{ categories (}\mathcal{A}\text{-small)}$$

Not quite:

$$U \xrightarrow{i} V \xrightarrow{j} W \quad i^* j^* \mathcal{F} \neq (j \circ i)^* \mathcal{F}$$

$$\underbrace{\quad \quad \quad}_{j \circ i}$$

$$\alpha_{ij} : i^* j^* \Rightarrow (j \circ i)^*$$

not iso of functors.

$$f: X \rightarrow Y \quad f^{-1} \mathcal{F}(V) = \lim_{\leftarrow U \supset V} \mathcal{F}(U)$$

$$\{ (X, U) \mid U \in \mathcal{F}(U) \}$$

$$f^{-1} \text{ adj } f_*$$

$$u \xrightarrow{i} v \xrightarrow{j} w \xrightarrow{k} z$$

$$\begin{array}{ccc}
 i^*(j^*(k^* \mathcal{F})) & \xrightarrow{\alpha_{i,j}(k^* \mathcal{F})} & (ji)^*(k^* \mathcal{F}) \\
 \downarrow i^*(\alpha_{j,k} \mathcal{F}) & \curvearrowright & \downarrow \alpha_{ji,k}(\mathcal{F}) \\
 i^*((kj)^* \mathcal{F}) & \xrightarrow{\alpha_{i,kj} \mathcal{F}} & (kji)^* \mathcal{F}
 \end{array}$$

Def A pseudofunctor  $\mathcal{X}: \mathcal{C} \rightarrow \underline{\text{Cat}}$   
(or say 2-catgory)

is a rule  $\mathcal{X}: \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\underline{\text{Cat}})$   
i.e. catgories

•  $\mathcal{X}: \text{Hom}_{\mathcal{C}}(a,b) \rightarrow \text{Fun}(\mathcal{X}a, \mathcal{X}b)$

together w/ for all composable arrows  
 $a \xrightarrow{f} b \xrightarrow{g} c$   
in  $\mathcal{C}$

a natural trans.

$$\mathcal{X}_{f,g} \equiv \alpha_{f,g}: \mathcal{X}(g) \mathcal{X}(f) \rightarrow \mathcal{X}(gf)$$

in  $\text{Fun}(\mathcal{X}a, \mathcal{X}c)$

s.t.  $\forall a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$

we have  $\alpha_{gh,h} (\mathcal{X}h \circ \alpha_{f,g}) = \alpha_{f,hg} (\alpha_{g,h} \circ \mathcal{X}f)$

$\emptyset$  and  $\mathcal{X}(id) = id$ .

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ex:  $\text{Shv}_{\text{mX}}$   $X = \text{top space}$ .

$X$  a scheme,  $\text{Sch}_{\text{mX}}$  Zariski top

$\text{Sch}_{\text{mX}}(U) = \text{Objects}$   $\begin{matrix} Y \\ \downarrow \\ U \end{matrix}$  morphism of schemes

and  $\text{Homs} = \text{morphisms of } U\text{-schemes}$

i.e.  $\begin{matrix} Y & \xrightarrow{f} & Y' \\ & \searrow & \downarrow \\ & & U \end{matrix}$   $\text{Gmb}$

$$U \xrightarrow{i} V$$

$$i^* = \text{Sch}_{\text{mX}}(i) : \text{Sch}_{\text{mX}}(V) \rightarrow \text{Sch}_{\text{mX}}(U)$$

$$\begin{matrix} Y \times_U U & \rightarrow & Y & \longrightarrow & Y \times_V U \\ \downarrow & & \downarrow & & \downarrow \\ U & \dashrightarrow & V & & U \end{matrix}$$

$$j^* i^* = (ij)^*$$

$$D \otimes_B B \otimes_c A$$

$$\cong D \otimes_c A$$

Ex: Ringed

$\mathcal{C} = \text{Top spaces}$

$$\text{Ringed} : \mathcal{C}^{\text{op}} \rightarrow \underline{\text{Cat}}$$

Ringed  $(X) = \text{Category of ringed spaces on } X.$   
i.e. cat of sheaves of rings on  $X$ .

objects: sheaves of rings

morphisms: morph. of sheaves of rings

$$\begin{array}{ccc} X \xrightarrow{f} Y & \text{Ringed}(Y) & \xrightarrow{f^* = \text{Ringed}(f)} \text{Ringed}(X) \\ & \mathcal{O}_Y & \longmapsto f^{-1}\mathcal{O}_Y \end{array}$$

Similarly, have a substack LRinged

i.e. LRinged  $(X)$  is a subcat of Ringed  $(X)$  at  $X$

Rem: stalks  $(f^{-1}\mathcal{F})_x = \mathcal{F}_{f(x)}$  so  $f^{-1}$  (loc. ringed) is loc. ringed.

$$\begin{array}{l} \text{if } \mathcal{F} \text{ a sheaf on } X \\ s \in \mathcal{F}(U) \quad v \xrightarrow{i} U \\ s|_v = \mathcal{F}(i)(s) = \text{res}_{u \rightarrow v}(s) \\ = i^*s \end{array}$$

Def If  $\mathcal{C}$  is a site, a presheaf on  $\mathcal{C}$  is a pseudofunctor  $\mathcal{F}: \mathcal{C}^{op} \rightarrow \underline{\text{Cat}}$  site = cat + Groth. top.

Suppose  $\{u_i \rightarrow u\}_{i \in I}$  a cover of  $u \in \text{ob}(\mathcal{C})$

Rethine  $\text{Desc}(\mathcal{F}, \{u_i \rightarrow u\}) = \text{Glue}(\mathcal{F}, \{u_i \rightarrow u\})$

objects: pairs  $((x_i)_{i \in I}, (\varphi_{ij})_{i,j \in I})$  where  $x_i \in \text{ob } \mathcal{F}(u_i)$

Notation:

$$u_{ij} = u_i \cap u_j$$

$$u_{ijk} = u_i \cap u_j \cap u_k$$

s.t.  $\varphi_{ij}: x_i|_{u_{ij}} \xrightarrow{\sim} x_j|_{u_{ij}}$  iso.

$$\mathcal{F}(u_i \cap u_j \rightarrow u_i)(x_i)$$

$$\varphi_{jk}|_{u_{ijk}} \cdot \varphi_{ij}|_{u_{ijk}} = \varphi_{ik}|_{u_{ijk}}$$

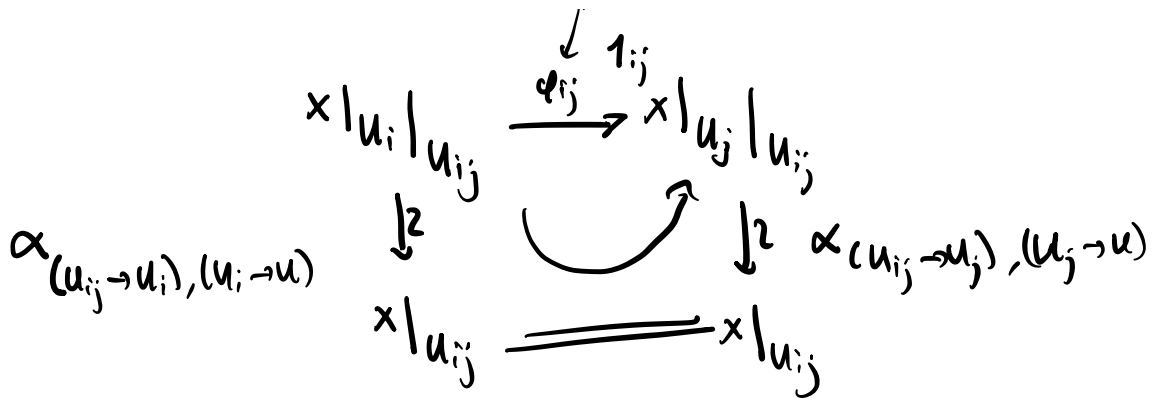
$$\text{Hom}_{\mathcal{F}(u_{ijk})} (x_i|_{u_{ijk}}, x_k|_{u_{ijk}})$$

We say  $\mathcal{F}$  is a stack if the canonical map

$$\mathcal{F}(u) \rightarrow \text{Desc}(\mathcal{F}, \{u_i \rightarrow u\}_{i \in I}) \text{ is an equiv of cats for } \mathcal{C} \text{ covers.}$$

$$\downarrow \psi$$

$$x \longmapsto ((x|_{u_i}), (\varphi_{ij}))$$



Ex:  $\text{Shv}_X$      $\text{Sch}_X$      $\text{Rngd}$      $\text{LRngd}$

Also shaves!

$$\begin{array}{ccc}
 \underline{\text{Set}} & \longrightarrow & \underline{\text{Cat}} \\
 \underline{S} & \longrightarrow & \underline{S}
 \end{array}$$

$$\begin{aligned}
 \text{ob}(S) &= S \\
 \text{Hom}_S(a,b) &= \begin{cases} \{id_a\} & \text{if } a=b \\ \emptyset & \text{else} \end{cases}
 \end{aligned}$$

if  $\mathcal{F}$  is a presheaf on  $\mathcal{C}$   
 then via above, get  $\text{spresheaf}$ .

$$\mathcal{F}^{\text{st}}: \mathcal{C}^{\text{op}} \longrightarrow \underline{\text{Set}} \longrightarrow \underline{\text{Cat}}$$

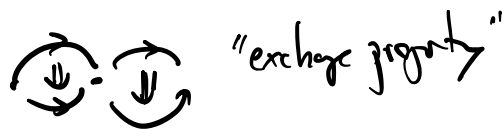
$\alpha$ 's for ps. track  
 all identities.

then  $\mathcal{F}^{\text{st}}$  is a stack  $\iff \mathcal{F}$  is a sheaf.

Moral: Stack  $\iff$  glueable stuff.

Maps between stacks. (preserves fibers)  
 presheaves

$$\mathcal{C} \xrightarrow{\mathcal{X}, \mathcal{Y}} \underline{\mathcal{C}}_t$$



$f: \mathcal{X} \rightarrow \mathcal{Y}$  is a choice for each  $u \in \mathcal{C}$  of a fiber

$$f(u): \mathcal{X}(u) \rightarrow \mathcal{Y}(u)$$

$$u \xrightarrow{\alpha} v$$

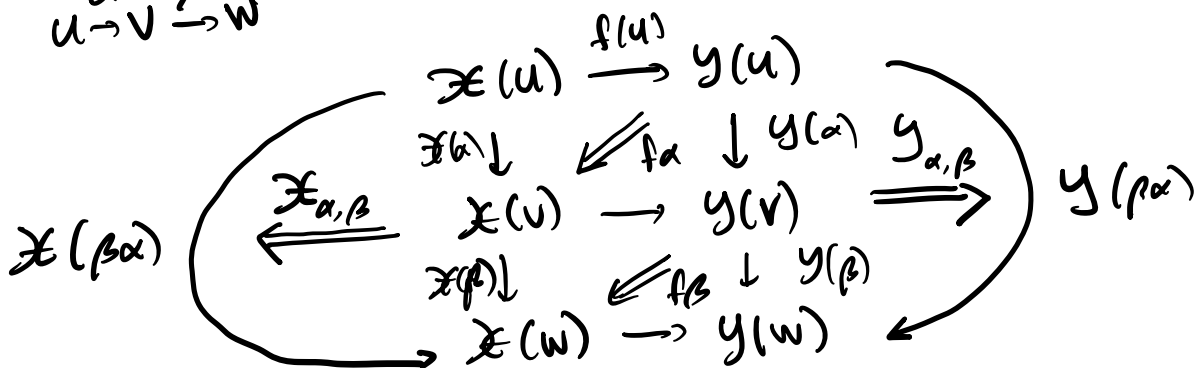
$$\mathcal{X}(u) \xrightarrow{f(u)} \mathcal{Y}(u)$$

$$\mathcal{X}(u) \downarrow \swarrow f(\alpha) \downarrow \mathcal{Y}(u)$$

$$\mathcal{X}(v) \xrightarrow{f(v)} \mathcal{Y}(v)$$

s.t.

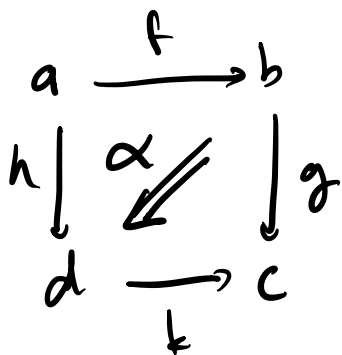
$$u \xrightarrow{\alpha} v \xrightarrow{\beta} w$$



$$\mathcal{X}(u) \xrightarrow{f(u)} \mathcal{Y}(u)$$

$$\mathcal{X}(u) \downarrow \swarrow f(\beta\alpha) \downarrow \mathcal{Y}(u)$$

$$\mathcal{X}(w) \xrightarrow{f(w)} \mathcal{Y}(w)$$



shorthand for

$$\alpha \in \text{Nat}(gf, kh)$$

if  $f, g: X \rightarrow Y$

$\alpha: f \Rightarrow g$

for each  $u \in \text{nat } X$

$$\begin{array}{ccc}
 & f(u) & \\
 \alpha(u) & \xrightarrow{\quad} & g(u) \\
 & \downarrow \alpha_u & \\
 & g(u) &
 \end{array}$$



# Yoneda Games

Top - site of top games.

$\text{Shv}_{\text{Top}}$  stack of all shvs on all sites.

suppose  $X \in \text{Ob}(\text{Top})$

$$X \rightsquigarrow \text{Shv}(X)$$

Consider  $h_X$  rep functor.

$$h_X(Y) = \text{Hom}_{\text{Top}}(Y, X) \quad \text{this is a sheaf. (and so a stack)}$$

$$\{U_i \rightarrow U\} \text{ cov in top.} \quad \text{Hom}(U, X) \rightarrow \prod \text{Hom}(U_i, X) \rightrightarrows \prod \text{Hom}(U_j, X)$$

$$\underline{\text{Hom}}_{\text{stacks Top}} \text{ ps. funts.} (h_X, \text{Shaf}_{\text{Top}}) \overset{\cong}{\underset{\substack{\uparrow \\ \text{eq. of cats}}}{\text{}}} \text{Shaf}(X)$$

Stacks Yoneda  $\text{Hom}_{\text{Fun}(\mathcal{C}^{\text{op}}, \text{Sets})} (h_X, \mathbb{F}) = \mathbb{F}(X)$