Roughly: a stack ~ sheaf of categories

ex: X top space $\qquad \mathrm{Shv}_X : \mathrm{Open}(X)^{op} \longrightarrow \mathrm{Cat}$

$$U \longmapsto \mathrm{Shv}(U)$$

given $U \overset{i}{\hookrightarrow} V$, $\mathcal{F} \in \mathrm{Shv}(V)$, can consider $i^{-1}\mathcal{F} \in \mathrm{Shv}(U)$

$$\mathrm{Shv}_X(i) : \mathrm{Shv}(V) \xrightarrow{i^{-1}} \mathrm{Shv}(U)$$

Q: is this a presheaf of Cat's?

$\underline{\mathrm{Cat}}$ = Cat. of ($\mathcal{U}$-small) categories

Not quite: $\quad U \overset{i}{\hookrightarrow} V \overset{j}{\hookrightarrow} W \qquad i^{-1}j^{-1}\mathcal{F} \neq (j\circ i)^{-1}\mathcal{F}$

$j \circ i$

$\alpha_{i,j} : i^{-1}j^{-1} \Longrightarrow (j\circ i)^{-1}$

nat iso of functors.

$f : X \to Y \qquad f^{-1}\mathcal{F}(V) = \varinjlim_{U \supset V} \mathcal{F}(U)$

$\mathcal{F} \qquad \{(x, f) \mid f \in \mathcal{F}(U)\}$

f^{-1} adj f_*

$$u \xrightarrow{i} v \xrightarrow{j} w \xrightarrow{k} z$$

$$\begin{tikzcd}
i^*(j^*(k^*\mathcal{F})) \arrow[r, "\alpha_{i,j}(k^*\mathcal{F})"] \arrow[d, "i^*(\alpha_{j,k}\mathcal{F})"'] \arrow[dr, phantom, "\circlearrowright"] & (ji)^*(k^*\mathcal{F}) \arrow[d, "\alpha_{ji,k}(\mathcal{F})"] \\
i^*((kj)^*\mathcal{F}) \arrow[r, "\alpha_{i,kj}\mathcal{F}"'] & (kji)^*\mathcal{F}
\end{tikzcd}$$

<u>**Def**</u> A pseudofunctor $\mathcal{X} : \underset{\text{cat}}{\mathcal{C}} \longrightarrow \underline{\mathrm{Cat}}$ (or any 2-category)

is a rule $\mathcal{X} : \mathrm{Ob}(\mathcal{C}) \longrightarrow \mathrm{Ob}(\underline{\mathrm{Cat}})$
i.e. categories

- $\mathcal{X} : \mathrm{Hom}_{\mathcal{C}}(a,b) \longrightarrow \mathrm{Fun}(\mathcal{X}a, \mathcal{X}b)$

together w/ for all composable arrows $a \xrightarrow{f} b \xrightarrow{g} c$ in $\mathcal{C}$

a natural trans.
$$\mathcal{X}_{f,g} \equiv \alpha_{f,g} : \mathcal{X}(g)\mathcal{X}(f) \to \mathcal{X}(gf)$$
in $\mathrm{Fun}(\mathcal{X}a, \mathcal{X}c)$

s.t. $\forall$ $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$

we have $\alpha_{gh,h}(\mathcal{X}h \circ \alpha_{f,g}) = \alpha_{f,hg}(\alpha_{g,h} \cdot \mathcal{X}f)$

• and $\mathcal{F}(id) = id$.

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ex: $\underset{\sim}{Shv}_X$ $\quad X$ a top space.

X a scheme, $\underset{\sim}{Sch}_X$ Zariski top

$\underset{\sim}{Sch}_X(U) =$ Objects $\begin{matrix} Y \\ \downarrow \\ U \end{matrix}$ morphism of schemes

and Homs = morphisms of U-schemes

i.e.
$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr] & & Y' \arrow[dl] \\ & U & \end{tikzcd}$$
commutes

$U \xrightarrow{i} V$

$i^* = \underset{\sim}{Sch}_X(i) : \underset{\sim}{Sch}_X(V) \to \underset{\sim}{Sch}_X(U)$

$$\begin{tikzcd} Y \times_V U \arrow[r] \arrow[d] & Y \arrow[d] \arrow[r, mapsto] & Y \times_V U \arrow[d] \\ U \arrow[r, dashed] & V & U \end{tikzcd}$$

$j^* i^* = (ij)^*$ $\qquad D \otimes_B B \otimes_C A$

$\cong D \otimes_C A$

Ex: Ringed $\qquad\qquad \mathcal{C} = $ Top spaces

$\underset{\sim}{\text{Rnged}} : \mathcal{C}^{op} \longrightarrow \underline{\text{Cat}}$

$\underset{\sim}{\text{Rnged}}(X) = $ Category of rnged spaces on X.

i.e. cat of sheaves of rngs on X.

objects: sheaves of rngs

morphisms: morph. of sheaves of rngs

$$X \xrightarrow{f} Y \qquad \underset{\sim}{\text{Rnged}}(Y) \xrightarrow{f^* = \underset{\sim}{\text{Rnged}}(f)} \underset{\sim}{\text{Rnged}}(X)$$

$$\mathcal{O}_Y \longmapsto f^{-1}\mathcal{O}_Y$$

Similarly, have a substack $\underset{\sim}{\text{LRnged}}$

i.e. $\underset{\sim}{\text{LRnged}}(X)$ is a subcat of $\underset{\sim}{\text{Rnged}}(X)$ at X

Rem: stalks $\quad (f^{-1}\mathcal{F})_x = \mathcal{F}_{f(x)}$ so f^{-1}(loc. rnged) is loc. rnged.

$$\boxed{\begin{array}{l} \text{if } \mathcal{F} \text{ a sheaf on } X \\ s \in \mathcal{F}(U) \quad V \xrightarrow{i} U \\ s|_V = \mathcal{F}(i)(s) = res_{U,V}(s) \\ \qquad = i^* s \end{array}}$$

Def If $\mathcal{C}$ is a site, a prestack on $\mathcal{C}$ is a pseudofunctor

$\mathcal{X} : \mathcal{C}^{op} \longrightarrow \underline{Cat}$

site = cat + Groth. top.

Suppose $\{U_i \longrightarrow U\}_{i \in I}$ a cover of $U \in ob(\mathcal{C})$

Define $Desc(\mathcal{X}, \{U_i \to U\}) = Glue(\mathcal{X}, \{U_i \to U\})$

<u>objects</u>: pairs $((x_i)_{i\in I}, (\varphi_{ij})_{i,j \in I})$

where $x_i \in ob\, \mathcal{X}(U_i)$

Notation:

$U_{ij} = U_i \cap U_j$

$U_{ijk} = U_i \cap U_j \cap U_k$

s.t. $\varphi_{ij} : x_i|_{U_{ij}} \xrightarrow{\sim} x_j|_{U_{ij}}$ iso.

$x_i|_{U_{ij}} = \mathcal{X}(U_i \cap U_j \to U_i)(x_i)$

&

$\varphi_{jk}|_{U_{ijk}} \cdot \varphi_{ij}|_{U_{ijk}} = \varphi_{ik}|_{U_{ijk}}$

$\mathrm{Hom}_{\mathcal{X}(U_{ijk})}(x_i|_{U_{ijk}}, x_k|_{U_{ijk}})$

We say $\mathcal{X}$ is a stack if the canonical map

$\mathcal{X}(U) \longrightarrow Desc(\mathcal{X}, \{U_i \to U\}_{i\in I})$ is an equiv of cats for all covers.

$x \longmapsto ((x|_{U_i}), (\varphi_{ij}))$

1_{ij}

$$\begin{tikzcd} X|_{U_i}|_{U_{ij}} \arrow[r, "\varphi_{ij}"] \arrow[d, "\wr"', "\alpha_{(U_{ij}\to U_i),(U_i\to U)}"'] & X|_{U_j}|_{U_{ij}} \arrow[d, "\wr\ \alpha_{(U_{ij}\to U_j),(U_j\to U)}"] \\ X|_{U_{ij}} \arrow[r, equal] & X|_{U_{ij}} \end{tikzcd}$$

<u>Ex:</u> $\underset{\sim}{Shv}_X$ $\quad$ $\underset{\sim}{Sch}_X$ $\quad$ <u>Ringed</u> $\quad$ $\underset{\sim}{\text{LRinged}}$

<u>Also sheaves!</u>

$$\underline{Set} \longrightarrow \underline{Cat}$$

$$S \longrightarrow \underline{S} \qquad ob(\underline{S}) = S$$

$$\mathrm{Hom}_{\underline{S}}(a,b) = \begin{cases} \{id_a\} & \text{if } a=b \\ \emptyset & \text{else} \end{cases}$$

if $\mathcal{F}$ is a presheaf on $\mathcal{C}$

then via above, get a prestack.

$$\mathcal{F}^{st}: \mathcal{C}^{op} \to \underline{Set} \longrightarrow \underline{Cat}$$

α's for ps. funct all identities.

then $\mathcal{F}^{st}$ is a stack $\Leftrightarrow$ $\mathcal{F}$ is a sheaf.

<u>Moral:</u> Stack $\longleftrightarrow$ gluable stuff.

Homs between Stacks. (pseudo functors)
prestacks

$\mathcal{C} \xrightarrow{\mathcal{X},\mathcal{Y}} \underline{\mathcal{C}at}$

"exchange property"

$f:\mathcal{X} \to \mathcal{Y}$ is a choice for each $U \in \mathcal{C}$ of a functor

$f(U): \mathcal{X}(U) \to \mathcal{Y}(U)$

$U \xrightarrow{\alpha} V$

$$\begin{tikzcd}
\mathcal{X}(U) \arrow[r, "f(U)"] \arrow[d, "\mathcal{X}(\alpha)"'] \arrow[dr, phantom, "\swarrow f(\alpha)"] & \mathcal{Y}(U) \arrow[d, "\mathcal{Y}(\alpha)"] \\
\mathcal{X}(V) \arrow[r, "f(V)"'] & \mathcal{Y}(V)
\end{tikzcd}$$

s.t.

$U \xrightarrow{\alpha} V \xrightarrow{\beta} W$

$$\begin{tikzcd}
\mathcal{X}(U) \arrow[r, "f(U)"] \arrow[d, "\mathcal{X}(\alpha)"'] \arrow[dd, bend right=80, "\mathcal{X}(\beta\alpha)"'] \arrow[dr, phantom, "\swarrow f\alpha"] & \mathcal{Y}(U) \arrow[d, "\mathcal{Y}(\alpha)"] \arrow[dd, bend left=80, "\mathcal{Y}(\beta\alpha)"] \\
\mathcal{X}(V) \arrow[r] \arrow[d, "\mathcal{X}(\beta)"'] \arrow[dr, phantom, "\swarrow f\beta"] & \mathcal{Y}(V) \arrow[d, "\mathcal{Y}(\beta)"] \\
\mathcal{X}(W) \arrow[r] & \mathcal{Y}(W)
\end{tikzcd}$$

$\mathcal{X}_{\alpha,\beta} \Leftarrow \qquad \Rightarrow \mathcal{Y}_{\alpha,\beta}$

$$\begin{tikzcd}
\mathcal{X}(U) \arrow[r, "f(U)"] \arrow[d, "\mathcal{X}(\beta\alpha)"'] \arrow[dr, phantom, "\swarrow f(\beta\alpha)"] & \mathcal{Y}(U) \arrow[d, "\mathcal{Y}(\beta\alpha)"] \\
\mathcal{X}(W) \arrow[r, "f(W)"'] & \mathcal{Y}(W)
\end{tikzcd}$$

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "h"'] & b \arrow[d, "g"] \arrow[dl, phantom, "\alpha\ \swarrow"] \\
d \arrow[r, "k"'] & c
\end{tikzcd}
$$

shorthand for

$\alpha \in \mathrm{Nat}(gf, kh)$

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if $f, g : \mathcal{X} \longrightarrow \mathcal{Y}$ $\quad \alpha : f \Rightarrow g$

for each u nat trans

$$
\begin{tikzcd}
\mathcal{X}(u) \arrow[r, "f(u)"] \arrow[r, bend right=60, "g(u)"'] \arrow[r, phantom, shift right=3ex, "\Downarrow \alpha_u"] & \mathcal{Y}(u)
\end{tikzcd}
$$

Yoneda Games

Top - Site of top spaces. $\underset{\sim\sim}{\text{Shv}}_{Top}$ stack of all shvs on all spaces.

suppose $X \in Ob(Top)$ $X \rightsquigarrow Shv(X)$

Consider h_X rep functor.

$$h_X(Y) = Hom_{top}(Y, X)$$ this is a sheaf. (and so a stack)

$\{U_i \to U\}$ cov in top. $Hom(U,X) \to \prod Hom(U_i, X) \rightrightarrows \prod Hom(U_{ij}, X)$

$$\underline{Hom}_{\substack{\text{Stacks}/Top \\ \text{ps. functs.}}}(h_X, Shv_{Top}) \simeq Sheaf(X)$$

eq of cats

Standard Yoneda $Hom_{Fun(\mathcal{C}^{op}, Sets)}(h_X, \mathcal{F}) = \mathcal{F}(X)$