Maps to projective space (following Eisenbud & Harris)

A ring characterize maps $\mathrm{Hom}_{A\,\mathrm{sch}}(X, \mathbb{P}^n_A)$

$\mathbb{P}^n_A = \mathrm{Proj}\, A[x_0, \dots, x_n]$

Case $X = \mathrm{Spec}\, K$ a field, $(A \to K)$

$$\mathrm{Proj}\, A[\vec{x}] = \bigcup_i \mathrm{Spec}\, \underbrace{A[x_i/x_j]_j}_{A[\vec{x}]_{(x_j)}}$$

EH, Ex III-10

"Recall" below gives a bijection between $\mathfrak{p} \in \mathrm{Proj}\, A[\vec{x}]$ not containing f and $\mathrm{Spec}\, A[\vec{x}]_{(f)}$

$(\mathfrak{q}\, A[\vec{x}]_f) \cap A[\vec{x}]$ $\mathfrak{p} \in \mathrm{Proj}\, A[\vec{x}]$ (hom prime ideals)

$\downarrow$ if $f \notin \mathfrak{p}$

$\mathfrak{q}$ $(\mathfrak{p}\, A[\vec{x}]_f)_0 \in \mathrm{Spec}\, A[\vec{x}]_{(f)}$ deg 0 elements in $A[\vec{x}][f^{-1}]$

note: if $\mathfrak{p} \in \mathrm{Proj}\, A[\vec{x}]$ then $\mathfrak{p} \notin \mathrm{Spec}\, A[\vec{x}]_{(x_i)}$ only if $x_i \in \mathfrak{p}$.

$\Rightarrow \Big(\mathfrak{p} \notin \mathrm{Spec}\, A[\vec{x}]_{(x_i)}$ all $i \iff x_i \in \mathfrak{p}$ all i
$\Rightarrow$ irrel. ideal in $\mathfrak{p} \Big)$

can check:

$$\mathrm{Spec}\, A[\vec{x}]_{(x_i)} \cap \mathrm{Spec}\, A[\vec{x}]_{(x_j)} = \mathrm{Spec}\, A[\vec{x}]_{(x_i x_j)}$$

In particular, this describes $\operatorname{Proj} A[\vec{x}] = \mathbb{P}^n_A$ via gluing

$\operatorname{Spec} A[\vec{x}]_{(x_i)} = \mathbb{A}^{n,i}_A$

Suppose given $\varphi : \operatorname{Spec} K \longrightarrow \mathbb{P}^n_A$. know $\exists\, i$ s.t.

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "\varphi_i"] \arrow[dr, bend right, "\varphi"'] & \mathbb{A}^{n,i}_A \arrow[d] \\
 & \mathbb{P}^n_A
\end{tikzcd}$$

$$\varphi_i^{\#} : A\left[x_0/x_i,\ x_1/x_i, \ldots, x_i/x_i, \ldots, x_n/x_i\right] \longrightarrow K$$
$$x_j/x_i \longmapsto a_j$$

notice if image also lies in $\mathbb{A}^{n,k}_A$

$$\varphi_k^{\#} : A\left[x_0/x_k, \ldots, x_n/x_k\right] \longrightarrow K$$
$$x_j/x_k \longmapsto b_j$$

on overlap: $A\left[x_0/x_i, \ldots, x_n/x_i\right]\left[(x_k/x_i)^{-1}\right] \xrightarrow{x_j/x_i} a_j$

$$\| \qquad\qquad a_k = b_i^{-1} \uparrow \ \downarrow \cdot\ a_k^{-1} = b_i$$

$A[x_0, \ldots, x_n]_{(x_i x_k)}$

$$\|$$

$A\left[x_0/x_k, \ldots, x_n/x_k\right]\left[(x_i/x_k)^{-1}\right] \xrightarrow[x_j/x_k]{} b_j$

$$(a_0, a_1, \ldots, \underset{i\text{th}}{1}, \ldots, a_n) \underset{b_i}{\overset{a_k}{\rightleftarrows}} (b_0, b_1, \ldots, \underset{k\text{th}}{1}, \ldots, b_n)$$

i.e. these span same line in K^{n+1}

<u>Prop</u> get a bijection between lines in K^{n+1} & $\mathbb{P}^n_A(K)$ in this way.

$$\ell = \langle (\lambda_0, \ldots, \lambda_n) \rangle \quad \text{if } \lambda_i \neq 0 \in K,$$

consider $\left(\lambda_0/\lambda_i, \ldots, 1, \ldots, \lambda_n/\lambda_i \right)$

$$\cap$$

$$\mathrm{Hom}(\operatorname{Spec} K, \operatorname{Spec} A[\tfrac{x_0}{x_i}]_{(x_i)})$$

Discussion for local rings is almost the same. (replace $\neq 0$ by unit)

i.e. if B is a local A-algebra,

get a bijection $\mathbb{P}^n_A(B)$ and $\left\{ (b_0, \ldots, b_n) \in B^n \;\middle|\; \text{some } b_j \in B^* \right\} \Big/ \sim$

$[b_0 ; \ldots ; b_n]$

$\sim$ mult. by units in B^*

General B? $\cancel{[b_0 ; \ldots ; b_n]}$?

<u>Perspective:</u> $\ell \subset K^{n+1} \longleftrightarrow$ hyperplane $H \subset (K^{n+1})^*$

dual space spanned by linear functions on K^{n+1}

$= x_0, \dots, x_n$

$A[x_0, \dots, x_n]$ fcns on $\mathbb{A}^{n+1}_A$

<u>Def</u> define $\widetilde{\mathbb{P}}^n_A$: $\underline{A\text{-alg}s} \longrightarrow \underline{\text{Sets}}$

$$\widetilde{\mathbb{P}}^n_A(B) = \left\{ W \subset B^{n+1} \text{ submodules} \;\middle|\; B^{n+1}/W \text{ is rk 1 projective} \right\}$$

<u>Reminder/refresher</u>

R ring. P/R projective $\iff$ any surjection $M \twoheadrightarrow P$ splits

$\iff P \oplus Q \simeq R^N$ some Q (P f.gen)

$\overset{\text{f.g.}}{\iff}$ P loc free ($P \otimes_R R_\mathfrak{p} \simeq R_\mathfrak{p}^m$ some m all $\mathfrak{p} \in \operatorname{Spec} R$)

m = rank of P

if R is connected (i.e. no nontriv. idempotents) then m = constant over $\mathfrak{p}$

P f.g. is projective $\iff \forall \mathfrak{p} \in \operatorname{Spec} R$

$P \otimes_R R_\mathfrak{p} \simeq R_\mathfrak{p}^m$ some m.

Def An R-module P is <u>invertible</u> if it is rank 1 projective.

Note: if M any R-module, define $M^* = \mathrm{Hom}_R(M,R)$

$$M \otimes M^* \longrightarrow R$$

$$M \otimes_R \mathrm{Hom}(M,R)$$

$$m \otimes f \longmapsto f(m)$$

if M is invertible then this is an <u>iso</u>!

$$P \otimes_R \mathrm{Hom}(P,R) \longrightarrow R \qquad (P \text{ finitely presented})$$

$$\otimes_R R_{\mathfrak{p}}$$

$$\left(P \otimes_R \mathrm{Hom}(P,R)\right) \otimes_R R_{\mathfrak{p}} \longrightarrow R_{\mathfrak{p}}$$

$$\|$$

$P \otimes_R R_{\mathfrak{p}} \cong R_{\mathfrak{p}}$ $\quad \left(P \otimes_R R_{\mathfrak{p}} \right) \otimes_{R_{\mathfrak{p}}} (\quad)$

$$R_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \mathrm{Hom}(R_{\mathfrak{p}}, R_{\mathfrak{p}}) \longrightarrow R_{\mathfrak{p}}$$

$$r \otimes f \longmapsto f(r)$$

$$\mathbb{P}^n_A(B) = \{ W < B^{n+1} \mid B^{n+1}/W \text{ invertible} \}$$

$$= \{ 0 \to W \to B^{n+1} \to P \to 0 \mid P \text{ invertible} \} \big/ \sim$$

$$\begin{tikzcd} 0 \arrow[r] & W \arrow[r] \arrow[d, "\wr"] & B^{n+1} \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & W' \arrow[r] & B^{n+1} \arrow[r] & P' \arrow[r] & 0 \end{tikzcd}$$

$$= \{ B^{n+1} \twoheadrightarrow P \mid P \text{ invertible} \} \big/ \sim$$

$$\begin{tikzcd} B^n \arrow[r, two heads] \arrow[dr, two heads] & P \arrow[d, "\wr"] \\ & P' \end{tikzcd}$$

$$R_{\mathfrak{p}} \xrightarrow{f_{\mathfrak{p}}} \mathrm{Hom}_P(P,P)_{\mathfrak{p}} = \mathrm{Hom}_{R_{\mathfrak{p}}}(P_{\mathfrak{p}}, P_{\mathfrak{p}}) = R_{\mathfrak{p}}$$

$$\mathrm{Hom}_R(P,P) \simeq R \qquad R \xrightarrow{f} \mathrm{Hom}_P(P,P)$$

locally an iso.

$$= \{ B^{n+1} \twoheadrightarrow P \mid P \text{ invertible} \} \big/ \sim$$

$\to P$, $\to P'$ $\quad P \not\simeq P'$ then not $\sim$

if $B^{n+1} \twoheadrightarrow P$, $B^{n+1} \twoheadrightarrow P'$ and $P' \underset{\not\simeq}{\Rightarrow} P$ then

$$B^{n+1} \twoheadrightarrow P' \sim B^{n+1} \twoheadrightarrow P' \xrightarrow[\varphi]{\sim} P$$

$$\begin{tikzcd} & P \\ B^{n+1} \arrow[ur, "\sim"] \arrow[r, two heads] & P' \arrow[u, "\varphi"'] \end{tikzcd}$$

Q: $\begin{tikzcd} B^{n+1} \arrow[r, "\varphi"] \arrow[dr] & P \\ & P \arrow[u, "\rho \,\wr ?"'] \end{tikzcd}$ $\rho =$ mult. by $a \in B^\times$

$e_i \in B^{n+1}$

$\varphi(e_i) = p_i \in P$ $\quad$ pt $\leftrightarrow [p_0; \ldots; p_n]$

$\downarrow \cdot b \in B^\times$

$[p_0'; \ldots; p_n']$

Note: the SES $0 \to W \to B^{n+1} \to P \to 0$ $\underline{\text{splits}}$ since P is projective.

$\Rightarrow B^{n+1} \simeq W \oplus P$

$\quad\quad\quad$ rk n $\quad$ rk 1

Thm $h_{\mathbb{P}^n_A} = \widetilde{\mathbb{P}}^n_A$ on Zar top on A-alg

Pf idea: $\widetilde{\mathbb{P}}^n_A$ is a sheaf and locally agrees w/ $\mathbb{P}^n_A(-)$

$\square$ details in EH.

$\widetilde{\mathbb{P}}^n_A \in \mathrm{Shaf}_{\mathrm{Zar}(A\text{-alg})} \in \underset{\sim\sim}{\mathrm{Shv}}_{\mathrm{Zar}(\,)}(\mathrm{Spec}\,A)$

$\underset{\sim\sim}{\mathrm{Shv}}_{\mathrm{Zar}(A\text{-alg})}$ stck on $\mathrm{Zar}(A\mathrm{lg})$

<u>Globalize this:</u>

How do we compute $\mathrm{Hom}_{\mathrm{Sch}/A}(X, \mathbb{P}^n_A)$?

to do this, we note that $\widetilde{\mathbb{P}}^n_A$ extends uniquely as a sheaf to Sch/A

So - if we write any def of a sheaf on Sch/A which agrees w/ $\widetilde{\mathbb{P}}^n_A$ on $(A\text{-alg})^{op}$, we are victorious!

$$\widetilde{\mathbb{P}}^n_A(X) = \left\{ \mathcal{W} < \mathcal{O}_X^{n+1} \text{ subsheaf} \;\middle|\; \mathcal{O}_X^{n+1}/\mathcal{W} \text{ is loc. free rk } 1 \right\}$$

<u>Recall</u>: P/R loc. free rk n $\iff$ P/R proj. rk n

$\Updownarrow$

$\widetilde{P}/\mathcal{O}_{\mathrm{Spec}\,R}$ loc. free rk n $\mathcal{O}_{\mathrm{Spec}\,R}$-module.

P a loc. free $\mathcal{O}_X$ mod (~~f.gen~~ (f. presented))

$\iff \exists\, U_i$ cov X s.t. $P|_{U_i} \simeq \mathcal{O}_{U_i}^m$ $\iff$

$\forall Q \in X \quad P_Q \simeq \mathcal{O}_{X,Q}^m$

equ. $\quad \tilde{\mathbb{P}}^n_A(X) = \{ \mathcal{O}_X^{n+1} \twoheadrightarrow \mathcal{L} \mid \mathcal{L} \text{ inv. shf} \} / \sim$

if $\mathcal{L} \not\simeq \mathcal{L}'$ different

if $\mathcal{L} \simeq \mathcal{L}'$

reduce to both $\mathcal{L}$

use $\mathrm{Hom}(\mathcal{L}, \mathcal{L}) = \mathcal{O}_X(X)^*$

<u>next</u>: map $X \to \mathbb{P}^n$

$\longleftrightarrow$ inv shf $\mathcal{L}/X$

& global sections

$$\mathcal{O}_X^{n+1} \xrightarrow{s_0, \ldots, s_n} \mathcal{L}$$

$$e_i \longmapsto s_i$$