## Last time:

### Cartier Divisors:

$(U_i, f_i)$ $\{U_i\}$ cover of $X$ $(U_i,f_i)=(U_i,g_i)$

s.t. $f_i f_j^{-1} \in \mathcal{O}_X(U_i \cap U_j)^*$ $f_i \in \mathcal{K}(U_i)^*$ if $f_i = g_i u_i$ $u_i \in \mathcal{O}_X^*$

($X$ integral $\Rightarrow \mathcal{K}$ = constant shf $\kappa(X)$ = fr. field of $\mathcal{O}_X(U)$ $U$ affine open)

Imagine $(U_i, f_i) \rightsquigarrow$ zeros & poles of $f_i$

**Recall:** can form $\operatorname{div}(U_i, f_i) = \sum_{\substack{Z \subset X \\ \text{codim 1 irred closed}}} v_Z(f_i)\,[Z]$ $Z \cap U_i$

if $X$ is RICO "reg. in codim 1"

$\in \operatorname{WDiv}(X)$

if $\eta \in Z$ gen. point

$\mathcal{O}_{X,\eta}$ reg. loc. ring of dim 1

$\Rightarrow$ a disc. val ring.

We noticed, if $X$ is nice (integral, ...)
then any invertible sheaf $\mathcal{L}$ can be imbedded as
a submod of $\mathcal{K}_X$

Say $\mathcal{L}|_{U_i} = f_i \mathcal{O}_{U_i} \subset \mathcal{K}_{U_i}$ and $(U_i, f_i)$ is a Cartier Divis.

$U_i$ s.t. $\mathcal{L}|_{U_i} \simeq \mathcal{O}_{U_i}$

Conversely, if $(U_i, f_i)$ Cartier, can construct
invertible sheaf $\mathcal{L}|_{U_i} = f_i \mathcal{O}_{U_i}$ $\quad D \rightsquigarrow \mathcal{L}(-D)$

Gives a map
$$\begin{aligned} CDiv(X) &\longrightarrow Pic\, X \\ D &\longrightarrow \mathcal{L}(-D) \end{aligned}$$

<u>Def</u> $Pic\, X = \{$iso. classes of inv. sheaves$\}$
operation $[\mathcal{L}] + [\mathcal{M}] = [\mathcal{L} \otimes \mathcal{M}]$
is a group! $[\mathcal{O}_X] = id$ $\quad \mathcal{L}^* = \underline{Hom}(\mathcal{L}, \mathcal{O}_X)$
$$\mathcal{L}^* \otimes \mathcal{L} \simeq \mathcal{O}_X$$

<u>Defined</u> $(U_i, f_i) + (V_j, g_j) = (U_i \cap V_j, f_i|_{ij} g_j|_{ij})$
$(U_i, f_i) \sim (U_i, g_i)$ if $\exists\, h \in \mathcal{K}_X(X)^*$ s.t.
$$f_i = g_i h|_{U_i} \text{ all } i$$

**Thm** $\dfrac{\mathrm{CDiv}(X)}{\sim} = \mathrm{CaCl}(X) \xrightarrow{\sim} \mathrm{Pic}\, X$

(if all inv. shves are $\simeq$ to subshves of $\mathcal{K}_X$)

and if $X$ is integral Noeth, loc. factorial, separated then

$\mathrm{WDiv}\, X \simeq \mathrm{CDiv}\, X$ and $\mathrm{WCl}(X) \simeq \mathrm{CaCl}(X)$

(loc factorial = $\mathcal{O}_{X,x}$ is a UFD all $x \in X$)

if $f \in \mathcal{K}_X(X)^* = k(X)^*$, then consider

$$\mathrm{div}(f) = \sum_{\substack{Z \subset X \\ \text{codim 1 closed}}} v_Z(f)\,[Z] \in \mathrm{WDiv}(X)$$

$$\mathrm{WCl}(X) = \frac{\mathrm{WDiv}(X)}{\langle \mathrm{div}\, f \rangle_{f \in k(X)^*}}$$

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maps to $\mathbb{P}^n_A \longleftrightarrow$ line bundles w/ gen set of global sections

$\mathrm{WDiv} \underset{\text{lucky}}{=} \mathrm{CDiv} \rightsquigarrow$ line bundles.

Nice situation: $X$ Noeth, sep, integral, regular.

Given a divisor $D = \sum n_i [Z_i]$
Weil

$$\begin{array}{c} D = \sum n_i [Z_i] \\ \updownarrow \text{nice} \qquad n_i = v_{Z_i}(f_j) \\ (U_j, f_j) \\ \swarrow \qquad \searrow \\ \text{locally free } f_i \mathcal{O}_{U_i} = \mathcal{L}(-D) \qquad \mathcal{L}(D) = f_i^{-1} \mathcal{O}_{U_i} \end{array}$$

Prop

$$\frac{\Gamma(\mathcal{L}(D))}{\Gamma(\mathcal{O}_X(X))^*} = \{\text{effective divisors } D' \text{ s.t. } D' \underset{cl}{\sim} D\}$$

Def Weil div $\sum n_i [Z_i]$ is effective if $n_i \geq 0$ all $i$

Car. Div $(U_i, f_i)$ is effective if $f_i \in \mathcal{O}_X(U_i)$

Notation $D$ effective written $D \geq 0$.

Pf of prop: if $s \in \Gamma(\mathcal{L}(D))$ then $s|_{U_i} \in f_i^{-1} \mathcal{O}_X(U_i)$

u $\quad \Gamma(\mathcal{K}(X))^* = \mathcal{K}(X)^* \qquad s|_{U_i} = t_i f_i^{-1}$

$t_i \in \mathcal{O}_X(U_i)$

and $s = \frac{t_i}{f_i} = t_j / f_j$ on $U_i \cap U_j$

$f_i s = t_i \in \mathcal{O}_X(U_i)$

$\Rightarrow s \in f_i^{-1}\mathcal{O}_X(U_i)$

$\Rightarrow \quad (f_i, U_i) \sim (t_i, U_i)$ via mult. by $s$.

$\exists s \quad sf_i = t_i \qquad \geq 0$

map $\dfrac{\Gamma(X, \mathcal{L}(D))}{\Gamma(X, \mathcal{O}_X)^*} \longrightarrow \{\text{eff. cart. div } D' \sim D\}$

$\longleftarrow - - -$

and conversely $\square$

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<u>Recall</u>: characterized maps to proj. space via

$\mathrm{Hom}_A(X, \mathbb{P}^n) = \{ \mathcal{O}_X^{n+1} \twoheadrightarrow \mathcal{L} \} / \sim$

"lines in $A^{n+1}$" $\qquad \mathcal{L}$ loc. free rk 1

$\mathrm{Hom}_A(X, Gr(k,n)) = \{ \mathcal{O}^{n+1} \twoheadrightarrow \mathcal{M} \} / \sim$

$\mathcal{M}$ loc. free rk $k+1$

"$k+1$ planes in $A^{n+1}$"

$$\begin{tikzcd} \mathcal{O}^{n+1} \arrow[r, two heads] \arrow[dr, two heads] & \mathcal{M} \arrow[d, "?"] \\ & \mathcal{M}' \end{tikzcd}$$

Was this a waste of time?

Define for an A-alg $B$

$$\mathrm{RealProj}^n(B) = \left\{ B \overset{\text{im of}}{\hookrightarrow} B^n \right\}$$

$B \xrightarrow{\varphi} B'$ $\qquad B^n \supset \operatorname{im} f \quad f : B \hookrightarrow B^n$

$\qquad\qquad\qquad\qquad \rightsquigarrow \varphi$

$B'^n \supset (\operatorname{im} f \circ \varphi) \quad f \circ \varphi : B' \longrightarrow B'^n$

$k[x] \longrightarrow k$

$x \longmapsto 0$

$k[x] \xrightarrow{(0,x)} k[x]^2$

$g \longmapsto (0, xg)$

$(0, \text{cond})$

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So far: our basic machine is

functors: $(R\text{-alg})^{op} \longrightarrow \text{Sets}$

$(Sch_{/R}) \longrightarrow \text{Sets}$

and have Groth. top. which we place on $Sch_{/R}$ or $(R\text{-alg})^{op}$

(Big) Zariski top.

Want more topologies.

Reasons: Historical: difficult to express computations in classical alg geom via Zariski.

"étale top"

Different topologies do different things

étale top = great job for "tame" phenomena avoiding "inseparable" issues

to deal w/ characteristic

"flat" top. (fppf / fffp): works in all characteristics, occasionally horribly different than expected.

"other flat top" (fpqc): generally not used for computation, but useful for cdh Sing.

smooth, syntomic, h-top, Nisnevich top,
crystalline site.

<u>One thing we want</u>: if $X$ is a schem, that $\mathrm{Hom}(-, X)$ should be a shef.

"subcanonical"

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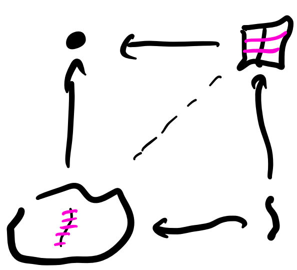
Basic ingredients for understanding étale, smooth..
lifting properties.

<u>Def</u> A map of rings $A \to B$ is formally smooth
if $\forall$ $A$-algs $C$, $I \triangleleft C$ s.t. $I^2 = 0$
and diagrams

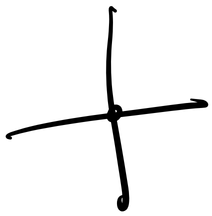


$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & C/I \end{tikzcd}$$

$\exists$ $B \to C$ s.t.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] \arrow[ur] & C/I \end{tikzcd}$$

commutes.



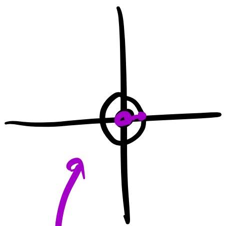
$$\frac{k[x,y]}{xy} \xrightarrow[\substack{x \mapsto 0 \\ y \mapsto 0}]{} k = \frac{k[\varepsilon]}{\varepsilon} \leftarrow \frac{k[\varepsilon]}{\varepsilon^2}$$

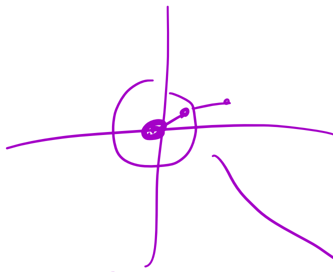
$x \mapsto \varepsilon + t\varepsilon^2$

$y \mapsto \varepsilon + s\varepsilon^2$

$\varepsilon^2$

$$\frac{k[\varepsilon]}{\varepsilon^3}$$





$\bullet = \frac{k[\varepsilon]}{\varepsilon}$

$\frac{k[\varepsilon]}{\varepsilon^2} =$

$\frac{k[\varepsilon]}{\varepsilon^3} =$