Newton's method

Given a poly f & an approx root (i.e. $f(a) \approx 0$) can "correct" it by solving

$$f(a+\varepsilon) \approx f(a) + \varepsilon f'(a) = 0$$

$$\varepsilon = -\frac{f(a)}{f'(a)} \qquad a' = a - \frac{f(a)}{f'(a)}$$

Algebraically: if we have a poly eqn "$f = 0$" over any A and an approx soln $a \in A$ in the sense that $f(a)^2 = 0$ and $f'(a) \in A^\times$ then in fact $a' = a - \frac{f(a)}{f'(a)}$ is a root in A!

[Note: if f poly, $\varepsilon^2 = 0$ $f(a+\varepsilon) = f(a) + \varepsilon f'(a)$]

(imagination $f' \neq 0$ is a proxy for smoothness)

Reminder S/R is formally smooth if $\forall$ rings A w/ $I \triangleleft A$, $I^2 = 0$ $\exists$ lifts in

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & A \arrow[d] \\
S \arrow[r] \arrow[ur, dashed] & A/I
\end{tikzcd}
\qquad a \mapsto \bar{a}$$

$$x \longmapsto \bar{a}$$

$$\left[S = R[x]/f \right]$$

Philosophically:

smooth = new problem

w/ Newton's method

$f' \neq 0$

$f(\bar{a}) = 0$

$\overline{f(a) = 0}$

$f(a) \in I \quad I^2 = 0$

i.e. we have noticed that if

$S = R[x]/f$ and roots of f have the prop. that $f'(a) \in R^*$

then smooth. S/R

$\frac{k[x]}{f}$ f no repeated roots

$\Downarrow$ smooth/ k.

$\vec{f}(\vec{a}+\vec{\varepsilon}) = \vec{f}(\vec{a}) + (D\vec{f})\vec{\varepsilon}$

$\left(\frac{\partial f_i}{\partial x_j}\right)$

all coords of $\vec{\varepsilon}$ are $\Delta = 0$

Alluding to: (formal statement later)

if "some Jacobian criterion" then formal smoothness

i.e. $S = R[\vec{x}]/\vec{f}$ w/ some conditions.

subplot: smoothness is naturally expressed in terms of poly rg presentations.

Def (A, I) is a Henselian pair (A rg $I \trianglelefteq R$) if for any poly f and $a \in A$ w/ $f(a) \in I$ $f'(a) \in A^\times$ then $\exists a' \in A$ w/ $f(a') = 0$ and

$a' + I = a + I$.

Rem: if $I^2 = 0$ then (R, I) is a Hens. pair.

"Prop" w/ some reasonable hypotheses, S/R form. smooth

$\iff$
$$\begin{tikzcd} R \arrow[r] \arrow[d] & A \arrow[d] \\ S \arrow[r] \arrow[ur, dashed] & A/I \end{tikzcd}$$
lifts for any (A, I) Henselian

and (A,I) Hens. pr $\iff$ $\forall$ S/R smooth

$\exists$ lifts
$$\begin{array}{ccc} R & \longrightarrow & A \\ \downarrow & \nearrow & \downarrow \\ S & \longrightarrow & A/I \end{array}$$

<u>Def</u> A local ring (R,m) is Henselian if R,m is a Hens pair.

<u>Silly examples</u>

$\mathbb{C}$ smooth over $\mathbb{C}$

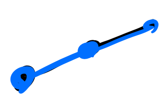
$$S = R = \frac{\mathbb{C}[x]}{x = f} \qquad A = \mathbb{C}[\varepsilon]/\varepsilon^2$$

$$x = \varepsilon \rightsquigarrow x = 0$$

$$f(\varepsilon)^2 = 0$$

$$S = \frac{\mathbb{C}[x]}{x^2} \qquad A = \mathbb{C}[\varepsilon]/\varepsilon^3$$

$S/\mathbb{C}$ not smooth.



Geometric sense of smoothness?

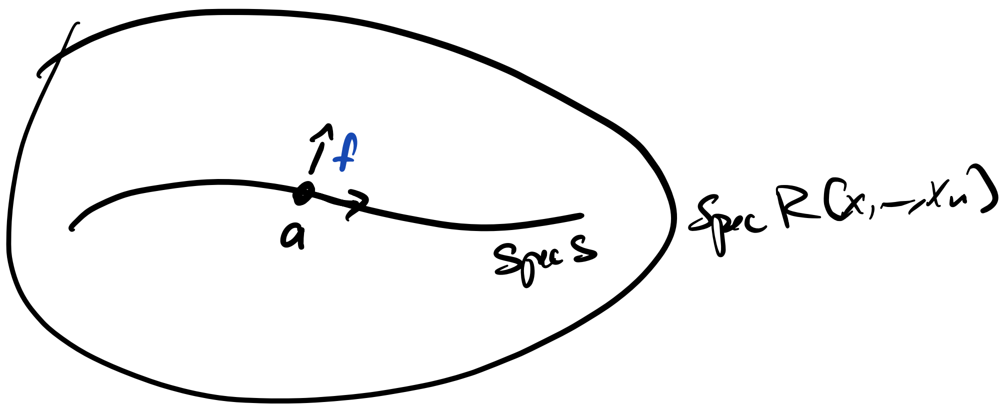
Idea: S/R smooth $\quad S = R[x_1,\dots,x_n]/J$

where near every point $(a_1,\dots,a_n)$

can find generators (near $\vec{a}$) $f_1,\dots,f_r$

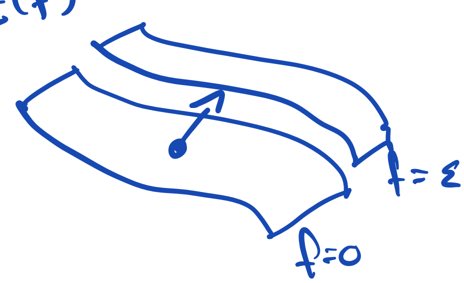
w/ $\dim S = \dim R + n - r$

and where $f_1,\dots,f_r$ "act like coords" at p



key to characteristic interpretation

$f \qquad Z(f)$



2 interpretations

of indep fcns cutting out of right #.

- regular sequences
- differentials

if $X \subset \mathbb{A}^n_k$ $\quad a \in \mathbb{A}^n_k(k)$

$a \in X(k)$

makes sense to ask if X cut out at a

by reg. sequence.

$$\mathcal{O}_{X,a} = \mathcal{O}_{\mathbb{A}^n,a} / (f_1, \dots, f_r)$$

$f_1, \dots, f_r$ indep in $\mathfrak{m}_a^{\mathbb{A}^n} / \mathfrak{m}_a^2$

Can ask for $\mathcal{O}_{X,a}$ reg local

all $a \in X$ scheme theoretic.

<u>Def</u> of X regular scheme

(Connection: $X \longrightarrow Y$ smooth, Y sm $\Rightarrow X$ regular)

Def $f: X \to Y$ smooth if f is loc. of finite presentation

by $\operatorname{Spec} S \to \operatorname{Spec} R$ S/R form.-smooth by extension.

Warning: if $X/_k$ field regular, $X \to \operatorname{Spec} k$ need not be smooth.

ex: $k = \operatorname{char} p$ $\quad \dfrac{k[x]}{x^p - a}$ is a field. $\Rightarrow$ regular.

$a \notin k^p$

but $L = \dfrac{k[x]}{x^p - a}$ not smooth.

$\bar{x} = \alpha \in L$ satisfies $\alpha^p = a$

Claim: formal that S/R smooth T/R any alg

$\Rightarrow$ $S \otimes_R T/_T$ smooth.

note $T = S = k[x]/_{x^p - a} = L$

$$S \otimes_R T = L \otimes_k L = \frac{L[x]}{x^p - a} = \frac{L[x]}{x^p - \alpha^p} = L[x]/(x - \alpha)^p$$

$x - \alpha = \varepsilon$

$= L[\varepsilon]/\varepsilon^p$

ex: $\left(k[x,\varepsilon]/\varepsilon^2\right) \Big/ \left(k[\varepsilon]/\varepsilon^2\right)$ smooth

$k[x]/k$ smooth

Def Given S/R, define $\Omega_{S/R}$

free S-module w/ generators da, $a \in S$ modulo the submod gen. by

- $d(a+b) - d(a) - d(b) \quad a, b \in S$
- $d(ab) - a\,d(b) - b\,d(a)$
- $dr = 0 \quad r \in R$

$\Omega_{S/R}$ is a repository for differentiation

$$S \xrightarrow{d} \Omega_{S/R}$$
$$a \longmapsto da$$

Def A derivation $S \xrightarrow{\varphi} M$ (M an S-mod)
R-

is a map which is an R-lin map & $\varphi(ab) = a\varphi(b) + b\varphi(a)$

$$\mathrm{Hom}_{S\text{-mod}}(\Omega_{S/R}, M) = \mathrm{Der}_R(S, M)$$

Given $B = A \oplus M$ M an A-module
w/ ring structure given by $(a+m)(b+n) = \underbrace{ab}_{A} + \underbrace{an+bm}_{M}$

$I = M$ square 0 ideal

$$S \xrightarrow{\psi} A \qquad R\text{-alg.}$$

$$S \xrightarrow{\tilde{\psi}} A \oplus M$$

$\tilde{\psi}(x) = \psi(x) + \psi'(x)$

$\psi' : S \longrightarrow M$

$$\tilde{\psi}(xy) = \psi(xy) + \psi'(xy)$$

$$\parallel$$

$$\tilde{\psi}(x)\tilde{\psi}(y) = (\psi(x) + \psi'(x))(\psi(y) + \psi'(y))$$

$$\psi'(xy) = \psi(x)\psi'(y) + \psi(y)\psi'(x)$$

ψ' is a derivation $S \longrightarrow M$ w/r to

S-mod structure on M via

$$S \xrightarrow[\psi]{} A \circlearrowright M$$

$$\left\{ \begin{matrix} \text{Extensions of } \psi \\ \text{to } A \oplus M \end{matrix} \right\} = \mathrm{Der}_{\psi}(S, M)$$

$$= \mathrm{Hom}_{\substack{S\text{-mod} \\ (\text{via } \psi)}}(\Omega_{S/R}, M)$$

Def S/R formally smooth if given $I \triangleleft A$, $I^2 = 0$ and a diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & A \arrow[d] \\ S \arrow[r] & A/I \end{tikzcd}$$

then $\exists\ S \to A$ s.t.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & A \arrow[d] \\ S \arrow[r] \arrow[ur] & A/I \end{tikzcd}$$

commutes

Def S/R formally étale if given $I \triangleleft A$, $I^2 = 0$ and a diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & A \arrow[d] \\ S \arrow[r] & A/I \end{tikzcd}$$

then $\exists!\ S \to A$ s.t.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & A \arrow[d] \\ S \arrow[r] \arrow[ur] & A/I \end{tikzcd}$$

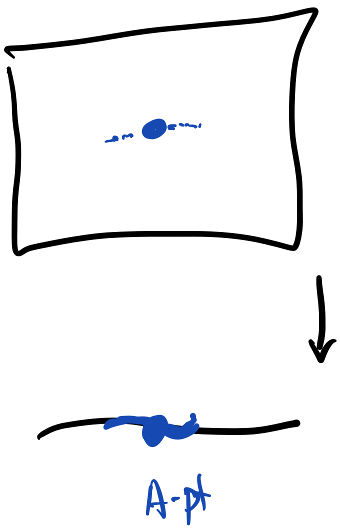
commutes

Def S/R formally unramified if given $I \triangleleft A$, $I^2 = 0$

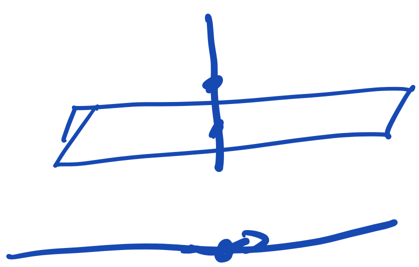
and a diagram
$$\begin{tikzcd} R \arrow[r] \arrow[d] & A \arrow[d] \\ S \arrow[r] & A/I \end{tikzcd}$$
then $\exists$ at most one $S \to A$

s.t.
$$\begin{tikzcd} R \arrow[r] \arrow[d] & A \arrow[d] \\ S \arrow[r] \arrow[ur] & A/I \end{tikzcd}$$
commutes

So: f. ét = f. sm + f. unram.



supposed to capture "local diffeomorphism"



Def smooth = f. sm + loc. finite presentation
(on affines)

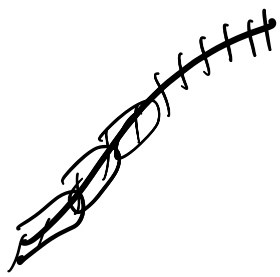
Def étale = f. ét + loc. finite presentation
(on affines)

Def unram = f. unr. + loc. finite type
(on affines)

(f. f. gen alg. relations..?)

$$\begin{tikzcd} W \arrow[r] \arrow[d, "\text{closed subsch}"'] & X \arrow[d, "\text{f-sm}"] \\ Z \arrow[r] \arrow[ur, dashed, "?"] & Y \end{tikzcd}$$

$\mathcal{I}_W^2 = 0$ in $\mathcal{O}_Z$



$$\begin{tikzcd} \operatorname{Spec} A/I \arrow[r] \arrow[d] & X \arrow[d, "f\text{-sm}"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

$I^2 = 0$