From last time:

<u>Idea of scheme</u>: "Blueprint" for getting sets of points $\updownarrow$ "solns to systems of equs" or something similar.

Started base ring R for our "equs"

a scheme S should associate to any R-algebra A, a set of "pts" $S(A)$ = "set of solns to our equs in A"

<u>Def</u>: R-space = functor $\underline{R\text{-alg}} \longrightarrow \underline{\text{Sets}}$

For any R-alg B $\quad S_B \equiv \left(S_B(A) = \mathrm{Hom}_{R\text{-alg}}(B,A) \right)$

"Representable functors"

<u>Prop</u> an R-space is $\simeq$ to some S_B iff it $\simeq$ to one the form $A \longmapsto \{ (a_i) \in A^I \mid f_j(a_i) = 0 \text{ all } j \in J \}$

$f_j \in R[x_i]_{i \in I}$

<u>Def</u> A moduli problem is a space.

"<u>Def</u>" A schematic space is one which is locally an affine space.

<u>Wrong Answer</u>: gluing $\longleftrightarrow$ limits maybe co

sch. spaces = co limits of affines? $\longrightarrow$

<u>Plausibility</u>: if $X = U_1 \cup U_2$ top space
open

$$\begin{tikzcd} U_1 & \\ U_1 \cap U_2 \arrow[u] \arrow[r] & U_2 \end{tikzcd} \quad \text{co lim} \rightsquigarrow \text{ in top spaces.} \quad \begin{tikzcd} U_1 \arrow[r] & X \\ U_1 \cap U_2 \arrow[u] \arrow[r] & U_2 \arrow[u] \end{tikzcd}$$

<u>But</u>: if S is any space (functor) we have

<u>Ex</u>: $S = \varinjlim_{b \in S(B)} S_B$

$$\mathrm{Hom}(S_B, S) \overset{\text{Yoneda}}{=} S(B)$$

$$\begin{tikzcd} & S \\ S_B \arrow[ur] & \\ S_{B'} \arrow[u] \arrow[uur] & \end{tikzcd}$$

limit

Deeper problem: limits of functors $\neq$ limits of spaces geometrically

top spaces

$$\begin{tikzcd} U_1 \arrow[r] & X \\ U_1 \cap U_2 \arrow[u] \arrow[r] & U_2 \arrow[u] \end{tikzcd}$$

pushout but

$$\begin{tikzcd} \mathrm{Hom}(Y,U_1) \arrow[r] & \mathrm{Hom}(Y,X) \\ \mathrm{Hom}(Y,U_1\cap U_2) \arrow[u] \arrow[r] & \mathrm{Hom}(Y,U_2) \arrow[u] \end{tikzcd}$$

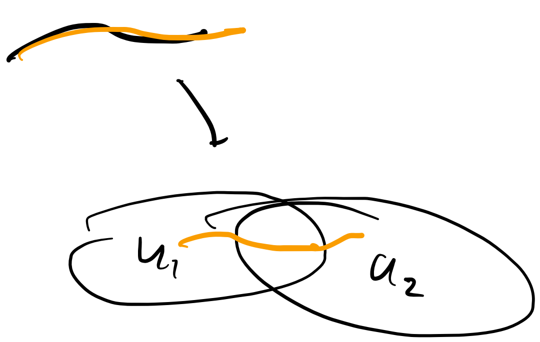
not necess. pushout.

$\mathcal{F}$ = pushout functor $\qquad \mathrm{Hom}(-,X)$

$$\mathcal{F}(Y) = \mathrm{Hom}(Y,U_1) \sqcup_{\mathrm{Hom}(Y,U_1\cap U_2)} \mathrm{Hom}(Y,U_2)$$

$$\downarrow$$

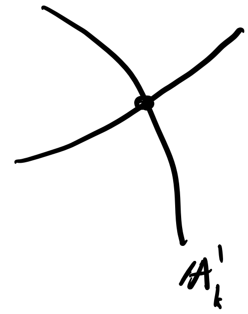
$$\mathrm{Hom}(Y,X)$$



$$\begin{tikzcd} k[x] \arrow[d] \\ k \end{tikzcd} \qquad \begin{tikzcd} x \arrow[d, mapsto] \\ 0 \end{tikzcd}$$

$$\begin{tikzcd} S_{k[x]} \arrow[r] & S \\ S_k \arrow[u] \arrow[r] & S(k[y]) \arrow[u] \end{tikzcd} \quad \text{pushout.}$$

$$0 \longleftarrow\!\shortmid\; y$$



Q: $S(k)$?

Q: $S(k[t])$?

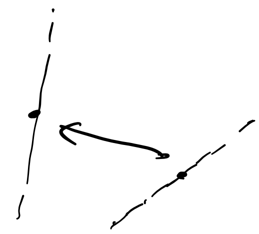
$$\underline{R\text{-alg}}^{op} \longrightarrow \underline{\text{Spaces}}$$
$$B \longmapsto S_B$$

Def A pushout of a diagram
$$\begin{tikzcd} A \\ B \arrow[u] \arrow[r] & C \end{tikzcd}$$
is the colim of this diagram – i.e. it's an object D w/ morphisms
$$\begin{tikzcd} A \arrow[r] & D \\ B \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$
(commute)

s.t. $\forall D'$ s.t.
$$\begin{tikzcd} A \arrow[r] & D' \\ B \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$
commutes $\exists !\ D \to D'$

s.t.
$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend left] & D \arrow[r] & D' \\ B \arrow[u] \arrow[r] & C \arrow[u] \arrow[ur] \end{tikzcd}$$
commutes.

$$\begin{tikzcd}
& k \arrow[d, phantom, "\rotatebox{90}{=}"] & \\
\mathbb{A}^1_k = & S_{k[x]}(k) \arrow[r] & S(k) \\
& S_k(k) \arrow[u, "\Gamma"] \arrow[r] & S_{k[y]}(k) \arrow[u] \\
& \text{pt} \arrow[u, phantom, "\rotatebox{90}{=}"] \arrow[r, bend right] & k \arrow[u, phantom, "\rotatebox{90}{=}"]
\end{tikzcd}$$

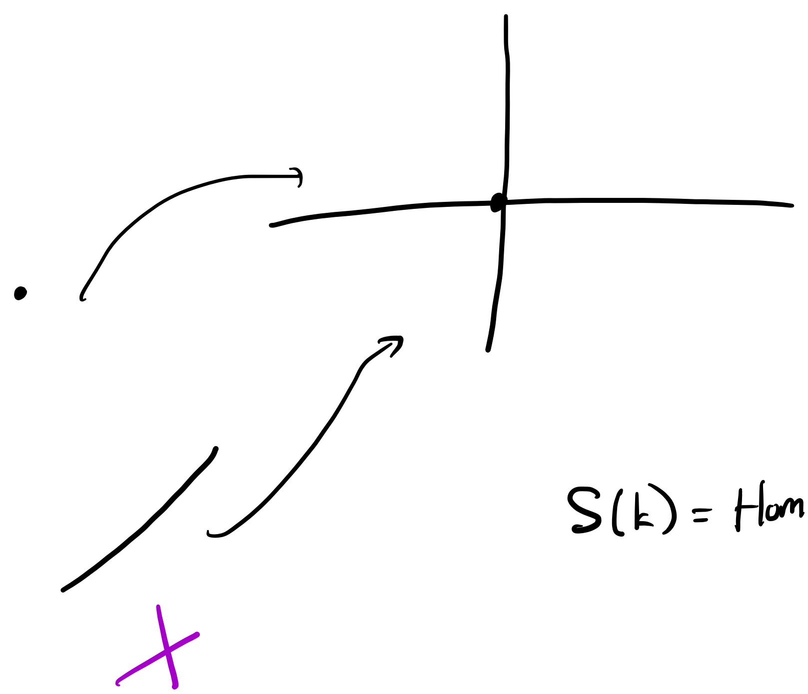


$$S_{k[x]}(k) = \mathrm{Hom}_{k\text{-alg}}(k[x], k) = k \text{ as a set}$$

$$\varphi \longmapsto \varphi(x)$$

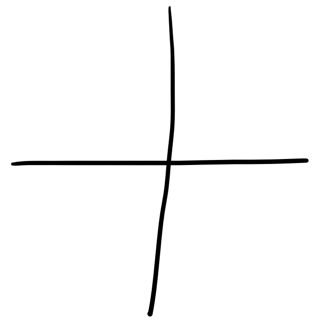
$$\text{pts} \longrightarrow \text{coords.}$$

$$S_k(k) = \mathrm{Hom}(k,k) = \{id\}$$



$$S(k) = \mathrm{Hom}(\underset{\text{"pt"}}{S_k}, S)$$

$$\begin{tikzcd} S_{k[x]} \arrow[r] & S_{\frac{k[x,y]}{xy}} \\ S_k \arrow[u] \arrow[r] & S_{k[y]} \arrow[u] \end{tikzcd}$$



$$\frac{k[x,y]}{xy}$$

$$\begin{tikzcd} k[x] \arrow[d] & \frac{k[x,y]}{xy} \arrow[l] \arrow[d] \\ k & k[y] \arrow[l] \end{tikzcd}$$

pullback square in rings

is a pushout in the subcat of spaces which are affine

Problem w/ these functors is non-locality.

Consider top spaces

$$\begin{tikzcd} U_1 \arrow[r] & X \\ U_1 \cap U_2 \arrow[u] \arrow[r] & U_2 \arrow[u] \end{tikzcd}$$

what is a map $Y \xrightarrow{f} X$ like?

gives a cover of $Y = V_1 \cup V_2 \quad V_i = f^{-1}(U_i)$

we find that critical feature is: maps $Y \to X$ are computed locally on Y.

i.e. $\mathrm{Hom}(Y, X) = \{ (f_1, f_2) \in \mathrm{Hom}(V_1, X) \times \mathrm{Hom}(V_2, X) \mid f_1|_{V_1 \cap V_2} = f_2|_{V_1 \cap V_2} \}$

This is an example of a sheaf.

__Def__ Let Y be a top space. $\mathrm{Open}(Y)$ = the category w/ objects $U \subset Y$ open and morphisms inclusions

A __presheaf__ $\mathcal{F}$ on Y is a functor $\mathrm{Open}(Y)^{op} \to \underline{\mathrm{Sets}}$

notation: if $f \in \mathcal{F}(U)$ and $i : V \to U$ inclusion we write $f|_V = \mathcal{F}(i)(f)$

A presheaf $\mathcal{F}$ is called a __sheaf__ if whenever U_i a cover of U then the natural map

$$\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$$
$$f \longmapsto (f|_{U_i})$$

gives a bijection

$$\mathcal{F}(U) \longleftrightarrow \{ (f_i) \in \prod \mathcal{F}(U_i) \mid f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \}$$

"things in $\mathcal{F}$ are defined locally"

Cat def: $\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$

is an equalizer diagram

<u>Most important observation</u>: if X, Y top spaces then

$$\begin{aligned} \mathrm{Open}(Y)^{op} &\longrightarrow \mathrm{Sets} \\ U &\longmapsto \mathrm{Hom}(U, X) \end{aligned}$$

is a <u>sheaf</u>. Very <u>important exercise</u>.

<u>Want to say</u>: functor $\mathrm{Hom}(-, X)$ is a sheaf.

<u>Def</u> if $\mathcal{X}$ is a subcat of Top spaces, s.t. contains all open inclusions then we say that a functor

$\mathcal{F} : \mathcal{X}^{op} \longrightarrow \mathrm{Sets}$ is a <u>sheaf</u> (on $\mathcal{X}$) an $\mathcal{X}$-sheaf

if $\forall$ spaces $X \in \mathcal{X}$ $\mathcal{F}|_{\mathrm{Open}(X)^{op}}$ is a sheaf.

X top space $U \subset X$ open subset $U \hookrightarrow X$ "Big topology" ex: $\mathcal{X}$ = all top spaces $\qquad Z$ top space

$$\mathcal{F}(X) = \mathrm{Hom}_{\mathrm{cont}}(X, Z)$$

this is a sheaf i.e. $\mathcal{F}|_{\mathrm{Open}(Y)^{op}}$ any Y

$$\mathrm{Open}(Y)^{op} \longrightarrow \mathrm{Sets}$$
$$U \longmapsto \mathrm{Hom}(U, Z) \quad \text{is a sheaf.}$$

More generally a sheaf on top sp. Y w/ values in a cat $\mathcal{C}$

is: a functor $\mathcal{F}: \mathrm{Open}(Y)^{op} \longrightarrow \mathcal{C}$ (presheaf in $\mathcal{C}$)

s.t. $\forall$ open covers U_i of U we have an eq. diagram

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

in particular these products & equalizers must exist in $\mathcal{C}$.