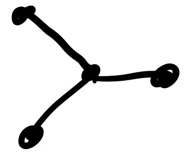
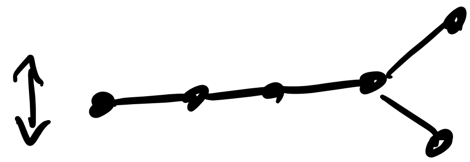


D_4



$PO_n \circlearrowright SO_n$

$SO_n/Z(SO_n)$

$GO_n/Z(\ )$

$O_n/Z(O_n)$

Concrete Interpretation for étale sheaves/gly.
(or flat topologies)

"<u>Faithfully flat descent</u>" DeMeyer-Ingraham "Separable algebras over comm. rings"
Knus-Ojanguren "Algèbres d'Azumaya et la théorie du descent"

(Milne EC, Tim Ford "Separable algebras", Murre lectures on the fund. gp)
FGA explained (Vistoli)

Suppose S/R faithfully flat extension of rings

(i.e. $0 \to M'' \to M \to M' \to 0$ exact iff
$0 \to M'' \otimes_R S \to M \otimes_R S \to M' \otimes_R S \to 0$ exact)

(example: if $(f_1, \dots, f_n) = R$ then $\prod_i R_{f_i}$ is faithfully flat over R)

__Theorem__ (f.f. descent) there's an equiv. of categories

$\{ R\text{-mods} \} \longrightarrow \{$ S-modules M & iso's $\varphi : (S \otimes_R S) \otimes_{i_1} M \to (S \otimes_R S) \otimes_{i_2} M$

s.t.

$$\begin{tikzcd}
(S \otimes_R S \otimes_R S) \otimes_{i_1} M \arrow[d, "\varphi_{12}"'] \arrow[dd, bend left=60, "\varphi_{13}"] \\
(S \otimes_R S \otimes_R S) \otimes_{i_2} M \arrow[d, "\varphi_{23}"'] \arrow[r, phantom, "\circlearrowright"] & {} \\
(S \otimes_R S \otimes_R S) \otimes_{i_3} M
\end{tikzcd}$$

$\}$

In case of $\prod R_{f_i} / R$ we're saying

R-mods $\longleftrightarrow$ M_i / R_{f_i} all i

& iso's $(M_i)_{R_{f_i f_j}} \xrightarrow{\varphi_{ij}} (M_j)_{R_{f_i f_j}}$

s.t.

$$
\begin{tikzcd}
(M_i)_{R_{f_i f_j f_k}} \arrow[r, "\varphi_{ij}"] \arrow[rr, bend right, "\varphi_{ijk}"'] & (M_j)_{R_{f_i f_j f_k}} \arrow[r, "\varphi_{ik}"] \arrow[d, phantom, "\circlearrowright"] & (M_k)_{R_{f_i f_j f_k}} \\
 & {} &
\end{tikzcd}
$$

$$\left(\prod R_{f_i}\right) \otimes_R \left(\prod R_{f_i}\right)$$

$$= \prod_{ij} R_{f_i f_j}$$

$$(M_i)_{R_{f_i f_j}} = M_i \otimes_{R_{f_i}} R_{f_i f_j}$$

$$= M_i \otimes_R R[f_j^{-1}]$$

$\cancel{M[f_i^{-1}]}$

$$= M_i \otimes_{R_{f_i}} R_{f_i}[f_j^{-1}]$$

$$\left(M_i \otimes_R R[f_j^{-1}]\right) \otimes_R R[f_k^{-1}]$$

$$= M_i \otimes_{R_{f_i}} R_{f_i f_j f_k}$$

Remark: S/R faithfully flat $\iff$ S/R flat & $\operatorname{Spec} S \to \operatorname{Spec} R$ is surjective.

ex: $k[x]/k$

ex: $\prod_{\mathfrak{p} \in \operatorname{Spec} R} R_{\mathfrak{p}} / R$ is surjective.

$$\begin{tikzcd} \prod R_{\mathfrak{p}} \arrow[d] & R \arrow[l] \arrow[d] \\ R_Q \arrow[r] & R/QR \end{tikzcd}$$

& if R is coherent $/R$ (f.g. ideals are f.p.) $\Rightarrow \prod$ flat = flat.

M/R coherent $\iff$ M f.g. & $\ker R^n \to M$ any morphism is f.g.

Def $\{U_i \xrightarrow{f_i} X\}$ is an fpqc cover if each f_i is q.compact & flat & surjective.

from above descent result:

eq. of cats

$\{$Q.coh $\mathcal{O}_X$-mods$\} \longleftrightarrow \{ M_i/\mathcal{O}_{U_i}$ q.coh & $\varphi_{ij}: M_i|_{U_i \times_X U_j} \to M_j|_{U_i \times_X U_j}$ s.t.

$ijk = U_i \times_X U_j \times_X U_k \qquad \varphi_{ik}|_{ijk} = \varphi_{jk}|_{ijk} \varphi_{ij}|_{ijk} \}$

via relative spec construction
can get an eq. of cat as above w/ affine morphisms
replacing q-coh sheaves (Munro)

Consequence (modulo work):

when we define cohomology w/ these different topologies

- Zariski $\quad U_i \to X$ open inclusions
- étale $\quad U_i \to X$ étale
- fpqc $\quad U_i \to X$ flat q-compact
- fppf $\quad U_i \to X$ flat. f. presented

in all cases $\quad R^n\Gamma_\tau(X, \mathcal{F}) = H^n(X_\tau, \mathcal{F})$

τ as above $\quad$ these all agree for $\mathcal{F}$ q-coherent.

computationally: for Zar, étale, fppf, nice notions of "points"
which let us reduce many computations to local ys in
these contexts

$\rightsquigarrow$ Structure of local rings:

- Zariski – local rings
- étale – Henselian local rings w/ sep. closed residue.

Recall: a geometric point of a scheme X is a morphism

$\operatorname{Spec} \Omega \xrightarrow{x} X$ $\quad \Omega$ alg. closed field.

$X \supset \operatorname{Spec} A$

$A \to A/\mathfrak{p} \hookrightarrow \operatorname{Frac} A/\mathfrak{p} \hookrightarrow \Omega$

$\mathcal{O}_{X_{\acute{e}t}, x}$ Henselian loc. ring w/ res. = sep closure of Frac $A/\mathfrak{p}$ in Ω

$\shortparallel$

$\mathcal{O}^{sh}_{X, \operatorname{im} x}$

Higher stacks

Recall: basic notion of stacks

X top space (or site)

$U \longmapsto \mathcal{X}(U)$ categories

$\cap$

X

pseudo functor

$U \xrightarrow{i} V \qquad \mathfrak{X}(V) \xrightarrow{i^*} \mathfrak{X}(U)$

s.t. coherence condition $\quad (g \circ f)^* \xrightarrow[\alpha_{f,g}]{\sim} f^* \circ g^*$

$$\begin{tikzcd}
 & (h \circ g \circ f)^* \arrow[dl] \arrow[dr] & \\
f^* \circ (h \circ g)^* \arrow[dr] & & (g \circ f)^* \circ h^* \arrow[dl] \\
 & f^* g^* h^* &
\end{tikzcd}$$

stack condition.

<u>Higher version</u>

$U \longmapsto \mathfrak{X}(U)$

$U \xrightarrow{f} V \longmapsto \mathfrak{X}(V) \xrightarrow{f^*} \mathfrak{X}(U)$

$$\begin{tikzcd}
\mathfrak{X}(U) \arrow[r] \arrow[rr, bend right] & \mathfrak{X}(V) \arrow[r] \arrow[d, phantom, "\Downarrow"] & \mathfrak{X}(W) \\
 & \; &
\end{tikzcd}$$

<u>gluing</u> higher stack condition

$x_i / U_i \qquad f_{ij} : x_i|_{ij} \longrightarrow x_j|_{ij}$

$x_i|_{ijk} \longrightarrow x_j|_{ijk} \longrightarrow x_k|_{ijk}$

$\cup \partial_{ijk}$

Alternately

Instead of thinking about (generalized) set-valued functors, consider functors into top spaces.

presheaf

$$X \supset U \rightsquigarrow \mathfrak{X}(U) \text{ top space.}$$

$V \subset U$

$$\begin{tikzcd} \mathfrak{X}(U) \arrow[r] \arrow[dr, bend right] & \mathfrak{X}(V) \arrow[d] \arrow[dl, phantom, "\Downarrow"] \\ & \mathfrak{X}(W) \end{tikzcd}$$

or often simplicial sets instead of top spaces.

Q: Given a functor

$$op(X)^{op} \longrightarrow SSets \text{ or } Top$$

what does it mean to be a sheaf?

Jardine ↙ $\leftarrow$ relate these ? $\rightarrow$ ↘ HAG perspective

Weak equivalences

" maps which become iso after sheafification

$$\mathfrak{X}(U) \to \prod \mathfrak{X}(U_i) \rightrightarrows \prod \mathfrak{X}(U_{ij}) \rightrightarrows$$

$$\mathfrak{X}(U) \to \operatorname{holim}(\text{diagram})$$

$\uparrow$ hom. equiv.

local hom. equivalences.

quotient localization.

$$\begin{tikzcd} \text{PreSheaf} \arrow[r, bend left, "\text{sheafify}"] & \text{Sheaf} \arrow[l, bend left, "\text{forget}"] \end{tikzcd}$$

<u>Plot:</u> Schemes

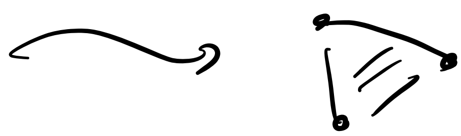
$$\mathrm{Hom}_{sch}(X, M) = \text{set}$$

M stack

$$\mathrm{Hom}_{stack}(X, M) = \text{Category (groupoid)}$$

$$\bullet \longrightarrow \bullet$$

M - 2stack



$M = K(n, A)$

$$\mathrm{Hom}_{\text{higher stack}}(X, M) = H^n(X, A)$$

$M = \underline{A}$ const shf 0-stack

$$\mathrm{Hom}(X, M) = \underline{A}(X) = H^0(X, A)$$

$$\Gamma(X, A)$$

Hom(X, M) simplicial

cosimplicial

$X_0 \leftleftarrows X_1 \Lleftarrow X_2$

$\mathrm{Hom}(X_0, M) \rightrightarrows \mathrm{Hom}(X_1, M) \rightrightarrows$

$X \rightsquigarrow$ "Spec R" R simplicial ring.

this gives us flexibility / language to construct more interesting / useful simplicial valued functors.

topl