Affine Schemes

Mantra: Every commutative ring is the ring of regular functions on a geometric space called an "affine scheme"

So: given a comm. ring R, points of this space: $\operatorname{Spec} R$
elements of R are functions on $\operatorname{Spec} R$

Recall: $\operatorname{Spec} R = \{\text{prime ideals in } R\}$

So if $\mathfrak{p} \in \operatorname{Spec} R$, $f \in R$ what is $f(\mathfrak{p})$

Recall: if $X = \mathbb{C}$-affine variety $R = \underset{A(X)}{\mathbb{C}[X]}$ ring of regular fcns

given $P \in X$ point, we have a map

$$m_p \longrightarrow \mathbb{C}[X] \xrightarrow{ev_p} \mathbb{C}$$
$$f \longmapsto f(P)$$

intrinsically, can identify $\mathbb{C} \simeq \frac{\mathbb{C}[X]}{m_p}$ (1st iso thm)

can. iso as) is a $\mathbb{C}$-v.space

So can identify $f(P)$ as the image of f under the canonical map $\mathbb{C}[X] \longrightarrow \mathbb{C}[X]/m_p$

More generally for $\mathfrak{p}$ prime in R, given $f \in R$
think of $f(\mathfrak{p})$ as the image of f in $R/\mathfrak{p} \subset \operatorname{frac}(R/\mathfrak{p})$

Ex: $R = \mathbb{Z}$ $\quad \mathfrak{p} = (5)$ $\quad f = 13$ $\quad f(\mathfrak{p}) \in \mathbb{Z}/5\mathbb{Z}$

$$f(\mathfrak{p}) = [13] = [3]$$

Ex: $R = \dfrac{\mathbb{C}[x,y,z]}{(xy - z^2)}$ $\quad \mathfrak{p} = (z,x)$ $\quad f = 3x^2 + 2y - xz$

identify $f(\mathfrak{p})$ & field in which it lives.

Remark: the zero ring exists & is an important example to keep in mind.

In this class: ring = unital, associative ring

ring 0 has a single element $0 = 1$

terminal object in ring

$$\text{Spec } 0 = \emptyset$$

Top Space: $\text{Spec } R = X$

Given $f \in R$, define $V(f) = \{ \mathfrak{p} \in \text{Spec } R \mid f(\mathfrak{p}) = 0 \}$ eqn in frac$(R/\mathfrak{p})$

$$= \{ \mathfrak{p} \in \text{Spec } R \mid f \in \mathfrak{p} \}$$

more generally, if $S \subset R$

$$V(S) = \bigcap_{f \in S} V(f)$$ these are the closed sets in the Zariski top

<u>Def</u> Zariski top on X is the top. whose closed sets are of the form $V(S)$.

$$V(S) \cap V(T) = V(S \cup T) \qquad V(1) = \emptyset$$

$$\bigcap_i V(S_i) = V(\bigcup_i S_i) \qquad V(0) = X$$

$$V(S) \cup V(T) = V(S \cdot T)$$

$\mathfrak{p}^{\cup}$ $\qquad S \subset \mathfrak{p} \Rightarrow \forall t \in T \;\; st \in \mathfrak{p}$

$$V(S) \subset V(S \cdot T)$$

$\mathfrak{p} \in V(S \cdot T)$ then either $T \subset \mathfrak{p} \Rightarrow \mathfrak{p} \in V(T) \Rightarrow \mathfrak{p} \in V(T) \cup V(S)$

or $\exists t \in T \setminus \mathfrak{p} \Rightarrow \forall s \in S \quad st \in ST \subset \mathfrak{p}$

$t \notin \mathfrak{p}$ prime

$\Rightarrow s \in \mathfrak{p}$ all $s \in S$

$\Rightarrow S \subset \mathfrak{p} \Rightarrow \mathfrak{p} \in \cup \ldots$

Closed sets are arb. $\bigcap$'s of $V(f)$'s.

$$V(f) \cup V(g) = V(fg)$$

$\Rightarrow$ complements $X_f = X \setminus V(f)$ are a basis for the open sets of the top

$X_f \leftrightarrow$ "basic open sets"

Remark $V(S) = V(\langle S \rangle) = V(\sqrt{\langle S \rangle})$

ideal gen by S

Feature: not all points are closed!

$\overline{\{\mathfrak{p}\}}$ not nesc. $\{\mathfrak{p}\}$

$$\overline{\{\mathfrak{p}\}} = \bigcap_{\mathfrak{p} \in V(I)} V(I) \qquad \mathfrak{p} \in V(I) \iff I \subset \mathfrak{p}$$

so $Q \in \bigcap_{\mathfrak{p} \in V(I)} V(I)$ means $\forall\, I$ s.t. $I \subset \mathfrak{p}$, have $I \subset Q$

$\Updownarrow$

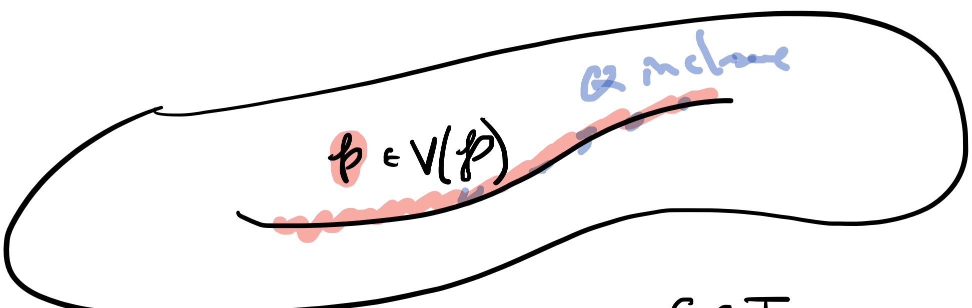
$\mathfrak{p} \in V(I) \Rightarrow Q \in V(I)$

i.e. $\mathfrak{p} \subset Q$

$$\boxed{\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})}$$

i.e. $\overline{\{\mathfrak{p}\}} = \{Q \in \operatorname{Spec} R \mid \mathfrak{p} \subset Q\}$

i.e. $\mathfrak{p}$ is closed $\iff$ it's maximal.



$S \subset T$

$V(S) \supset V(T)$

<u>Def</u> X a top space, $\mathcal{C}$ category (Sets, Ab.gps, Com.rngs, Rings, ...)

then a pre-sheaf on X w/ values in $\mathcal{C}$ is a functor

$$\mathcal{F} : \mathrm{Open}(X)^{op} \longrightarrow \mathcal{C}$$

Given a presheaf $\mathcal{F}$ on X, $p \in X$, we define

$$\mathcal{F}_p = \varinjlim_{U \ni p} \mathcal{F}(U)$$

i.e. the U's containing p form an inverse system

$$\begin{matrix} U & \\ & \leftarrow U \cap V \\ V & \end{matrix} \quad \text{"filtered system"}$$

i.e. the open sets containing p are a subcat of $\mathrm{Open}(X)$

s.t. given any two $U, V \in$ subcat, $\exists\, W \in$ subcat

w/ maps $W \to U$, $W \to V$ in subcat.

$\rightsquigarrow$ apply $\mathcal{F}$'s

$$\begin{matrix} \mathcal{F}(W) \leftarrow \mathcal{F}(U) \\ \nwarrow \mathcal{F}(V) \end{matrix}$$

diagram of objects in $\mathcal{C}$ which is cofiltered

can take $\varinjlim$ these.

Concretely: $\mathcal{F}_p = \dfrac{\{(U,f) \mid f \in \mathcal{F}(U),\ p \in U\}}{\sim}$

$(U,f) \sim (V,g)$ if $\exists\ \underset{\overset{\cup}{p}}{W} \subset U \cap V$

s.t. $f|_W = g|_W$

Def A presheaf $\mathcal{F}$ is a sheaf if $\forall\, U$ open $\{U_i\}$ cover of U, we have an equalizer diagram

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

$\mathcal{F}(\emptyset)$ cover by empty cover.

$$\prod_{\emptyset} \rightrightarrows \prod_{\emptyset}$$

$$\mathcal{F}(\emptyset) \simeq \underset{\text{terminal}}{\overset{\shortparallel}{}} \rightrightarrows \text{terminal}$$

ex: if $\mathcal{F}$ is a sheaf of rings $\mathcal{F}(\emptyset) = 0$ Ring

$\mathcal{F}$ is - - - Ab grps $\mathcal{F}(\emptyset) = 0$

$\mathcal{F}$ - - sets $\mathcal{F}(\emptyset) = \{*\}$

Prop (Sheafification) If $\mathcal{F}$ is a presheaf, $\exists$ a sheaf $\hat{\mathcal{F}}$ together w/ map of presheaves $\mathcal{F} \to \hat{\mathcal{F}}$ s.t. $\forall$ sheaves $\mathcal{G}$ and morphisms $\mathcal{F} \to \mathcal{G}$ $\exists!\ \hat{\mathcal{F}} \to \mathcal{G}$ s.t.

$$\begin{tikzcd} \mathcal{F} \arrow[rr] \arrow[dr] & & \mathcal{G} \\ & \hat{\mathcal{F}} \arrow[ur] & \end{tikzcd}$$ commutes.

(Adjointness: $\mathrm{Hom}_{\mathrm{pre}}(\mathcal{F}, \mathrm{forget}(\mathcal{G})) = \mathrm{Hom}_{\substack{\mathrm{shv} \\ \shortparallel \\ \mathrm{pres}}}(\hat{\mathcal{F}}, \mathcal{G})$)

$\boxed{\hat{\ } \dashv \mathrm{forget}}$

presheaves w/ values in Ab. gps are an Ab. cat.

"pointwise"

i.e. $f : \mathcal{F} \to \mathcal{G}$ presheaves

$(\ker f)(U) = \ker(f(U) : \mathcal{F}(U) \to \mathcal{G}(U))$

coker similarly

sheaves, not so much.

$(\ker f)(U) = \ker(f(U) : \mathcal{F}(U) \to \mathcal{G}(U))$

fine for shvs

define $(\mathrm{coker}' f)(U) = \mathrm{coker}(f(U) : \mathcal{F}(U) \to \mathcal{G}(U))$

$(\mathrm{im}' f)(U) \ldots$

not generally sheaves. $\mathrm{coker}\, f = \widehat{\mathrm{coker}' f}$

$\mathrm{im}\, f = \widehat{\mathrm{im}' f}$

Problem: sections of sheaves should be defined locally.

Prop: A morphism of sheaves of Ab. gps (w/values in any Ab cat)

is inj (surj) if $\forall$ $p \in X$

$\mathcal{F}_p \longrightarrow \mathcal{G}_p$ inj (surj)

$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact in shvs

$$\begin{array}{ccc} U & \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \\ \cup & \downarrow \qquad\quad \downarrow \\ V & \mathcal{F}(V) \longrightarrow \mathcal{G}(V) \end{array}$$

$\Leftrightarrow$ $0 \to \mathcal{F}'_p \to \mathcal{F}_p \to \mathcal{F}''_p \to 0$ exact $\forall P$.

$\boxed{\text{p. 15 in G. of Sch.}}$

Def $\mathcal{F} \to \mathcal{G}$ inj/surj if $\mathcal{F}_p \to \mathcal{G}_p$ is all P.

(rngs, sets, Ab gps...)

(loc) Rnged spcs