Open Immersions (= embedding)

Let X be a scheme. If $U \subset X$ open we say $(U, \mathcal{O}_X|_U)$ "$(X, \mathcal{O}_X)$" is an open subscheme of $X = (X, \mathcal{O}_X)$

We say the natural map $i : (U, \mathcal{O}_X|_U) \to (X, \mathcal{O}_X)$ is an open inclusion.

If $Y \xrightarrow{\varphi} X$ is a morphism of schemes, we say φ is an open immersion if $\exists\, U \subset X$ open s.t. φ factors as

$$\begin{tikzcd} Y \arrow[rr, "\varphi"] \arrow[dr, "\wr"'] & & X \\ & U \arrow[ur, "\text{open inclusion}"'] & \end{tikzcd}$$

Note: this is "local on X" i.e. if $Y \xrightarrow{\varphi} X$ a morphism and $\{U_i\}$ open cover of X then φ an open immersion $\iff$ $\forall i$, $\varphi^{-1}(U_i) \xrightarrow{\varphi} U_i$ is an open immersion.

Pop Quiz: Is this "local on Y"? if $Y \xrightarrow{\varphi} X$ morphism $\{V_i\}$ covers Y then φ open imm $\iff$ $\forall i$, $V_i \xrightarrow{\varphi|_{V_i}} X$ open imm?

$X \amalg X \longrightarrow X$ locally on $X \amalg X$ an open imm. not globally.

Gluing practice

Def $\mathbb{A}^1_k = \operatorname{Spec} k[x]$

Let $U_1 = \operatorname{Spec} k[x, x^{-1}] \underset{\text{open}}{\subset} \operatorname{Spec} k[x] = X_1$

$U_2 = \operatorname{Spec} k[y, y^{-1}] \subset \operatorname{Spec} k[y] = X_2$

Define: $\varphi : U_1 \xrightarrow{\sim} U_2$ by $\quad k[y, y^{-1}] \to k[x, x^{-1}]$, $\quad y \longmapsto x^{-1}$

Define $\underset{\underset{\mathbb{P}^1_k}{\shortparallel}}{X} = X_1 \sqcup_\varphi X_2$

Exercises:

1) For $a \in k$ when is the max'l ideal $(x-a) \in \operatorname{Spec} k[x] = X_1$ also in X_2? And in case it is, how is it represented as a max'l in $k[y]$?

2) For $f(x) \in k[x]$ irred poly, when is $(f(x)) \in \underset{\underset{X_1}{\shortparallel}}{\operatorname{Spec} k[x]}$ also in X_2? How do we represent it?

3) What are the points of $\mathbb{P}^1_k \setminus \underset{\underset{X_2}{\shortparallel}}{\operatorname{Spec} k[y]}$

Def If $X = \operatorname{Spec} R$, $I \trianglelefteq R$ we say that the map

$\operatorname{Spec} R/I \longrightarrow \operatorname{Spec} R$ (induced by $R \to R/I$)

is a closed inclusion and that $\operatorname{Spec} R/I$ is a closed subscheme of $\operatorname{Spec} R$.

We say $Y \xrightarrow{\varphi} \operatorname{Spec} R = X$ is a closed immersion if it factors

$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[dr, "\imath"'] & \operatorname{Spec} R \\ & \operatorname{Spec} R/I \arrow[u, "\text{imm.}"'] \end{tikzcd}$$

more generally, we say $\varphi : Y \longrightarrow X$ (X any scheme) is a closed immersion if $\forall$ covers $\{U_i\}$ of X w/ $U_i = \operatorname{Spec} R_i$ we have $\varphi^{-1}(U_i) \xrightarrow{\varphi} U_i$ are closed immersions.

equiv: $\exists$ cover $\{U_i\}$ s.t. ...

Lemmas for this to make any sense

- If $X = \operatorname{Spec} R$, $U = \operatorname{Spec} R_f \overset{\varphi}{\hookrightarrow} \operatorname{Spec} R$ (basic open) and $\imath : \operatorname{Spec} R/I \longrightarrow \operatorname{Spec} R$ then

$$\begin{tikzcd} \imath^{-1}(U) \arrow[r, "\imath"] \arrow[d, phantom, "\cong" rotate=90] & U \arrow[d, phantom, "\cong" rotate=90] \\ \operatorname{Spec} R_f/IR_f \arrow[r] & \operatorname{Spec} R_f \end{tikzcd}$$

and if $j : Y \longrightarrow \operatorname{Spec} R$ $\{U_i\}$ cov $U_i = \operatorname{Spec} R_{f_i}$

w/ $(f_i) = R$

and if
$$\begin{tikzcd} j^{-1}(U_i) \arrow[r] \arrow[d, phantom, "\cong"] & U_i \arrow[d, phantom, "\cong"] \\ \operatorname{Spec} R_{f_i}/I_i \arrow[r] & \operatorname{Spec} R_{f_i} \end{tikzcd}$$

then $\exists!\ I \triangleleft R$ s.t.
$$\begin{tikzcd} Y \arrow[rr] \arrow[dr, "\sim"'] & & \operatorname{Spec} R \\ & \operatorname{Spec} R/I \arrow[ur, "\text{imm}"'] & \end{tikzcd}$$

<u>translation of latter to</u> gsi given ideals $I_i \triangleleft R_{f_i}$

$$\begin{tikzcd} & \operatorname{Spec} R_{f_i}/I_i \arrow[r] \arrow[d, phantom, "\cong"] & \operatorname{Spec} R_{f_i} \arrow[d, phantom, "\cong"] \\ \operatorname{Spec} R_{f_if_j}/I_iR_{f_if_j} \arrow[ur, hook] \arrow[d, phantom, "\parallel"] & j^{-1}(U_i) \arrow[r] & U_i \\ j^{-1}(U_i \cap U_j) \arrow[ur] \arrow[dr] \arrow[rr] \arrow[d, phantom, "\parallel"] & & U_i \cap U_j = \operatorname{Spec} R_{f_if_j} \arrow[u] \arrow[d] \\ \operatorname{Spec} R_{f_if_j}/I_jR_{f_if_j} \arrow[dr, hook] & j^{-1}(U_j) \arrow[r] \arrow[d, phantom, "\cong"] & U_j \arrow[d, phantom, "\cong"] \\ & \operatorname{Spec} R_{f_j}/I_j \arrow[r] & \operatorname{Spec} R_{f_j} \end{tikzcd}$$

i.e. given $I_i \triangleleft R_{f_i}$ s.t. $I_i R_{f_if_j} = I_j R_{f_if_j}$

then $\exists!\ I \triangleleft R$ s.t. $IR_{f_i} = I_i$ $(f_i) = R$

Let $I = \{x \in R \mid x/1 \in I_i \text{ in } R_{f_i}\}$

$x \in I,\ r \in R \quad x/1 \in I \Rightarrow rx/1 = \frac{r}{1}\frac{x}{1} \in I \Rightarrow I \triangleleft R.$

by construction, $IR_{f_i} \subset I_i$

Need to show: $I_i \subset IR_{f_i}$

Let $y \in I_i$ WTS $y \in IR_{f_i}$ i.e. WTS $\exists x \in I$

$y = \frac{x}{f_i^{n_i}}$ some n_i or equiv. $y f_i^{n_i+m_i} = x f^{m_i}$ in R ($\in I$)

in R_{f_i} i.e. $y f_i^{N_i} = y'/1,\ y' \in I.$

by def. of I, this means $y f_i^{N_i}/1 \in I_j$ all j.

By def. of this criteria, set $N = \max\{N_i\}$ replace y by $f_i^N y$ any N.

i.e. can assume $y = y'/1,\ y' \in R.$

Let $J_j = \{r \in R \text{ s.t. } ry'/1 \in I_j\}$ $\quad J_j \triangleleft R$

and set $J = \bigcap_j J_j$ (i.e. $yJ \subset I$)

now $y = y'/1 \in I_i$ by hyp. so $J_i = R$

$y'/1 \in I_i R_{f_i f_j} = I_j R_{f_i f_j} \Rightarrow \exists n_j$ s.t.

$(f_i f_j)^{n_j} y' \in I_j \Rightarrow f_i^{n_j} y' \in I_j$ since $f_j \in R_{f_j}^*$

in R_{f_j}

so if $N = \max\{n_j\}$ then $f_i^N y' \in I_j$ all j

$\Rightarrow f_i^N \in J_j$ all j's $\Rightarrow f_i^N \in J$ $\square$.

$f_i^N y'/1 \in I_j$ all j

<u>Uniqueness?</u> If $I, I' \triangleleft R$ s.t. $IR_{f_i} = I'R_{f_i}$ all i.

WTS $I = I'$. Let $J = (I : I')$

$= \{r \in R \mid rI' \subset I\}$

Let $x \in I'$, $x/1 \in I'R_{f_i} = IR_{f_i} \Rightarrow \frac{x}{1} = y/f_i^{n_i}$

in R_{f_i} some $y \in I$

$\Rightarrow f_i^{n_i} f_i^{m_i} x = f_i^{m_i} \underset{\in I}{y}$ in R

$\Rightarrow f_i^{n_i + m_i} \in J$ let $N = \max\{n_i + m_i\}$

$\Rightarrow f_i^N \in J$ all i $\Rightarrow (f_i^N) \in J \Rightarrow J = R$

$\Rightarrow 1 \cdot I' \subset I \Rightarrow I' \subset I$. $\square$.

Better perspective: Ideal sheaves

Def if X a top space, $\mathcal{F}$ a sheaf of sets, $\mathcal{G}$ is a subsheaf of $\mathcal{F}$ if $\mathcal{G}(U) \subset \mathcal{F}(U)$ all U.

Def if $Y \xrightarrow{\varphi} X$ closed immersion then can define a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ (a subsheaf of $\mathcal{O}_X$ s.t. $\forall U$ $\mathcal{I}(U) \trianglelefteq \mathcal{O}_X(U)$)

such that $\forall U = \operatorname{Spec} R \subset X$ open affines

we have

$$\begin{array}{ccc} \varphi^{-1}(U) & \xrightarrow{\varphi} & U \\ \| & & \| \\ \operatorname{Spec} R/I & \longrightarrow & \operatorname{Spec} R \end{array}$$

we have $\mathcal{I}(U) = I \subset R = \mathcal{O}_X(U)$.

Know that since affines are a basis, $\mathcal{I}$ is determined by its behavior on affines.

$$\{\text{closed immersions}\} \longrightarrow \{\text{sheaf of ideals}\}$$

$$\overset{\longleftarrow}{?}$$

<u>Def</u> A sheaf of ideals $\mathcal{I} \lhd \mathcal{O}_X$ is <u>quasicoherent</u> if for any $\underset{\operatorname{Spec} R}{U} \subset X$ affine, $\exists\, I \lhd R$ ideal s.t. $\mathcal{I}(\operatorname{Spec} R_f) = I R_f$.

w/ identifications respecting restrictions

$$\begin{array}{ccc} \mathcal{I}(\operatorname{Spec} R_f) & \longrightarrow & \mathcal{I}(\operatorname{Spec} R_{fg}) \\ \| & & \| \\ I R_f & \xrightarrow[\text{canonical}]{} & I R_{fg} \end{array}$$

<u>Prov. def</u> subschemes $\longleftrightarrow$ q.coh ideal sheaves.

$$\operatorname{Spec} \mathbb{Q} \longrightarrow \operatorname{Spec} \mathbb{R}[x]$$

$$\mathbb{Q} \hookrightarrow \mathbb{R} \longleftarrow \mathbb{R}[x], \quad x \mapsto 0$$

$$\operatorname{Spec} \mathbb{R} \longrightarrow \operatorname{Spec} \mathbb{R}[x]$$