Math 6020, Graduate Algebra, Fall 2024, Homework 1

Instructor: Danny Krashen

Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. (a) Show that for a monoid M, given elements a, b, x such that $ax = 1$, $xb = 1$ we necessarily have $a = b$.

   (b) Show that this need need not hold if M is a general magma.

2. Recall that for a set X, we define $E_X = Map(X, X)$ to be the set of maps from X to itself and $S_X = Bij(X, X)$ to be the set of bijective maps from X to itself. As we saw in class, E_X is a monoid and S_X is a group.

   Let $G \subset E_X$ be a subset of mappings from X to X which forms a group under composition.

   (a) Give an example with $|G| \geq 2$ and $G \not\subset S_X$.

   (b) Show that if there exists $g \in G$ such that g is injective then $G \subset S_X$.

3. Suppose G is a finite group and p is the smallest prime number dividing the order of G. Show that if $H < G$ with $[G : H] = p$ then H is normal in G.

4. Suppose G is a group of order pq where p and q are distinct prime numbers. Show that G has subgroups $H, K < G$ of orders p and q respectively, and that every element g of G can be written uniquely in the form $g = hk$ for $h \in H$ and $k \in K$.

5. A 0-cube is a single point, and a 1-cube is a line segment.

   Inductively an n-cube is a n-dimensional polytope with $2n$ faces, each of which is an $(n-1)$-cube.

   Using the fact that each rotational symmetry of a face can be extended to a rotational symmetry of the entire n-cube, find the order of the symmetry group B_n of the n-cube.

6. Let G be a group and suppose that $g^2 = e$ for every $g \in G$. Show that G must be Abelian.

7. Suppose G is a finite group and $\phi : G \to G$ is an automorphism such that $\phi(g) = g$ if and only if $g = e$.

   (a) Show that every element of g may be written in the form $g = x^{-1}\phi(x)$ for some $x \in G$.

   (b) Show that if $\phi^2 = \mathrm{id}_G$ then G must be Abelian.

8. Let G be a finite group and suppose $m \mid |G|$ with $m > 1$. Show that the (set) map $G \to G$ given by $g \mapsto g^m$ is not surjective.