Math 6020, Graduate Algebra, Fall 2024, Homework 3

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Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. For a group G, note that the endomorphisms $End(G)$ of G (i.e., homomorphisms from G to itself) form a monoid with respect to composition.
   (a) For a cyclic group $C = \langle x \rangle$ of order n show that $End(C)$ is isomorphic to the (multiplicative) monoid $\mathbb{Z}/n\mathbb{Z}, \cdot)$.

   (b) Show that for a cyclic group C of order n, we have $Aut(C) \cong (\mathbb{Z}/n\mathbb{Z})^*$.

   (c) Show that if p and q are distinct primes with $p < q$ then there exists a nonabelian group of order pq if and only if $p \mid (q-1)$.

   (d) Show that in the previous case, when $p = 2$, there is a unique nonabelian group of order $2q$.

   (e) Show that if G is a group of order 30 with exactly 15 distinct 2-Sylow subgroups then G is isomorphic to the dihedral group $D_{15} = \langle \sigma, \tau | \sigma^{15} = e = \tau^2, \tau\sigma\tau = \sigma^{-1} \rangle$.

2. Show that if G is a group with Abelian normal subgroup N and suppose $H \subset C_G(N)$. Show that if the image $\overline{H}$ of H in G/N is cyclic then H is Abelian.

3. Let G be a finite group and P a p-Sylow subgroup.
   (a) Show that if $N_G(P) \triangleleft G$ then $N_G(P) = G$.

   (b) Show that $N_G(N_G(P)) = N_G(P)$.

4. Let G be a group and $N \triangleleft G$ a normal subgroup, and let $\pi : G \to \overline{G} = G/N$ be the canonical projection map. Choose any set-theoretic map $s : \overline{G} \to G$ such that $\pi \circ s = \mathrm{id}_{\overline{G}}$ and define a map

$$\phi : \overline{G} \to Aut(N)$$

   via $\phi(x)(n) = s(x)\ n\ (s(x))^{-1}$.
   (a) Show that ϕ is independent of the choice of s if and only if N is Abelian.

   (b) Show that if N is Abelian then ϕ is a group homomorphism.